Type Theory and Constructive Mathematics

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Type Theory with one universe (1972)

Terms $t, u, A, B \ ::= \ x \mid U \mid \lambda x.t \mid t\ u \mid (\Pi x : A)B$

$$\frac{}{\vdash} \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash}$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (\Pi x : A)B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash U}$$

$$\frac{\Gamma \vdash \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash t : (\Pi x : A)B \quad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(u)} \qquad \frac{\Gamma, x : A \vdash v : B}{\Gamma \vdash \lambda x.v : (\Pi x : A)B}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash B \quad A = B}{\Gamma \vdash t : B}$$

$$\frac{\Gamma \vdash A : U \quad \Gamma, x : A \vdash B : U}{\Gamma \vdash (\Pi x : A)B : U} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash A}$$

Automath

3 concepts from Automath

Context

Proof irrelevance

Definitional equality/book equality

Context

Notion of *context*

$x_1 : A_1,\ x_2 : A_2(x_1),\ x_3 : A_3(x_1, x_2), \ldots$

"let x be a natural number, assume that $\varphi(x)$ holds and let y be a rational number, ..."
$x : N,\ h : \varphi(x),\ y : Q, \ldots$

Compared to the usual mathematical notation, notice that we have an explicit name for the hypothesis that $\varphi(x)$ holds.

Context

$a_1, \ldots, a_n$ fits $x_1 : A_1, \ldots, x_n : A_n$ iff
$a_1 : A_1,\ a_2 : A_2(a_1),\ \ldots,\ a_n : A_n(a_1, \ldots, a_{n-1})$

This is what we have to use when we want to instantiate a statement starting with assertions of the form
"let x be a natural number, assume that $\varphi(x)$ holds and let y be a rational number, ..."

We give n and we check that $n : N$; then we give a proof $p : \varphi(n)$ and then we give $q : \mathbb{Q}$

n, p, q fits the context $x : N, z : \varphi(x), y : \mathbb{Q}$

Context

$\Gamma, \Delta, \ldots$ for context

The context lists the variables that are "alive" at a given point in a proof/definition. This is always finite.

Hypothetical judgements $\Delta \vdash a : A$

Interpretation between contexts $\Delta \rightarrow \Gamma$ if $\Gamma = x_1 : A_1, \ldots, x_n : A_n$

sequence $a_1, \ldots, a_n$ such that
$\Delta \vdash a_1 : A_1,\ \Delta \vdash a_2 : A_2(a_1),\ \ \ldots,\ \Delta \vdash a_n : A_n(a_1, \ldots, a_{n-1})$

Context

$$\frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda x.b : (\Pi x : A)B}$$

If we don't have context explicitely

$$\frac{b : B \quad [x : A]}{\lambda x.b : (\Pi x : A)B}$$

we have to state that the variable x should not occur free in any assumption of the derivation of $b : B$

Proof-irrelevance/subsets

The inverse function on $\mathbb{R}$ expects two arguments, one number $x : \mathbb{R}$ and one proof p that this number is apart from 0

However the result should depend only on the first argument. The second argument is "irrelevant" (but it is important that it is there)

Definitional equality

We use λ-terms to represent mathematical objects. We see $f = \lambda x.t$ as being a function defined by the equation $f\ a = t(a)$.

We consider

$$(\lambda x.t)\ a$$

and

$$t(a)$$

to be *definitionally* equal

This notion of definitional equality is specific to type theory (not mentionned in first-order logic). It is a "premathematical" notion

Definitional equality

This notion should be kept distinct from the notion of "book equality", which represents equality as a type $\mathrm{Id}_A\ a_0\ a_1$

We can build a proof that two elements $a_0\ \ a_1 : A$ are book equal by building a term $p : \mathrm{Id}_A\ a_0\ a_1$

We cannot *state* as a type that two elements are definitionally equal

Similarly we cannot state in the system that an element a is of type A (the statement $a : A$ is a judgement and not a proposition/type)

Martin-Löf tried 1979-1986 to identify definitional and book equality, but this has undesired consequences: type-checking=proof-checking becomes undecidable, and there is no model where types are interpreted as domains

Definitional equality

definitional equality is a notion which is unmentionable within the classical set theoretic framework

In MLTT, equality in the recursion equations

$$f(d,e)\ 0 = d \qquad\qquad f(d,e)\ (n+1) = e\ n\ (f(d,e)\ n)$$

is interpreted as definitional equality, just like the equality in explicit or abreviationary definitions like

$$\neg\ A = A \to N_0 : U$$

Logic of type theory

Stronger axioms for existence and disjunction

We can interpret the system HA^ω: quantification over functionals

We can interpret also $\mathsf{HA}^\omega + \mathsf{AC}$ since with strong existence we can interpret AC

$$(\forall x.\exists y.R(x,y)) \to \exists f.\forall x.R(x,f(x))$$

Proof-theoretical strength of type theory

With one universe, or even a sequence of universes, the system is *predicative*

We cannot interpret higher-order arithmetic (if consistent; all this follows from Gödel's incompleteness results)

The type $(\Pi X : U)(X \to X)$ is not of type U

The system $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{EM}$ is strictly stronger than any consistent predicative system

It follows that, if MLTT is consistent, there cannot be any syntactic translation of MLTT + EM into MLTT

Compare with the negative translation of $\mathsf{HA} + \mathsf{EM}$ in HA or of $\mathsf{HA}^\omega + \mathsf{EM}$ in HA^ω

Imprdicative/Predicative

This goes back to a famous debate between Poincaré and Russell

According to this discussion, the source of paradoxes was to be found in the "circular" quantification, e.g. when we quantify over all propositions and form a proposition

The usual example of impredicative definition is of the lower bound of a set of real numbers, where a real number is defined as a predicate over the rationals (Dedekind cut)

Here the type of propositions is represented by the universe U

Type Theory and Set Theory

What is a *set*?

It is a *tree*, via the correspondance membership = subtree

de Bruijn's version of Russell's paradox: define a tree to be normal if it is not equal to one of its subtree, then consider the collection of all normal trees and form a tree out of this collection. Is this tree normal?

Type Theory and Set Theory

If we have basic elements $0, 1, 2, \ldots$

The set $\{\{1, 1\}, \{2, 3\}\}$ should be the same as the set $\{\{3, 3, 2\}, \{1\}\}$

Equality is a bissimulation

The types of tree

A tree T is given inductively by a family of (sub)trees indexed by an arbitrary small type

sup X f : T whenever $X : U$ and $f : X \to T$

The type T is large: we do not have $T : U$

Equality of trees

$\sup\ X\ f =_T \sup\ Y\ g$ is defined inductively as

$$
\begin{array}{rl}
& (\Pi x : X)(\Sigma y : Y)\ f\ x\ =_T\ g\ y \\
\times & (\Pi y : Y)(\Sigma x : X)\ f\ x\ =_T\ g\ y
\end{array}
$$

and then $t \in \sup\ X\ f$ is defined as

$$(\Sigma x : X)\ t =_T f\ x$$

Interpretation of set theory

The interpretation of $\exists ! x . \forall y . \neg (y \epsilon x)$ is provable

This validates most axiom of set theory CZF (Constructive Zermelo-Fraenkel)

This interpretation is due to Peter Aczel (1978)

Interpretation of set theory

One can interpret ZF in CZF + EM

So if CZF is consistent there is no interpretation of CZF + EM in CZF

If we start from a type theory with $V : V$ we validate

$$\exists x.\forall y.(y \epsilon x \leftrightarrow \neg(y \epsilon y))$$

and hence we can interpret Russell's paradox

Interpretation of set theory

With $V : V$ we don't need the type of trees but we can use instead the type of pointed graphs

We define equality as bissimulation and interpret even non well-founded set theory!

We can interpret sets such as $a = \{a\}$ or $b = \{\{b\}, b\}$

Sets as pointed graphs

Define the equivalence $E : A \to B \to V$ between A, R, a and B, S, b as

$$\exists R.(\forall a_0\ a_1 : A.\forall b_1 : B.R\ a_0\ a_1 \to E\ a_1\ b_1 \to \exists b_0 : B.S\ b_0\ b_1 \wedge E\ a_0\ b_0)$$

$$\wedge\ \ (\forall b_0\ b_1 : B.\forall a_1 : A.S\ b_0\ b_1 \to E\ a_1\ b_1 \to \exists a_0 : A.R\ a_0\ a_1 \wedge E\ a_0\ b_0)$$

Type theory and set theory

Both interpretations illustrate well the differences between type theory and set theory

The objects of type theory have a direct computational interpretation

As a foundation, CZF is justified in term of MLTT

Bishop set theory

Hilbert and Brouwer had strong disagreements on foundation of mathematics

Both however agreed that most mathematics would be more difficult to develop in a constructive framework (without EM)

This was in large part proved wrong by Bishop 1967, who could develop basic results in analysis directly in a constructive framework

One can argue that some concepts/proofs are even more natural in a constructive framework, cf. Bridger and Stolzenberg work on the fundamental theorem of calculus

Bishop set theory

Bishop works in an informal set theory, which has a more direct computational interpretation than CZF

A set is defined when we describe how to construct its members . . . and describe what it means for two members to be equal

A *set* is a type with an equivalence relation

$A : U$ with $R : A \rightarrow A \rightarrow U$ with proof p that R is an equivalence relation

A set is of the form (A, R, p)

In particular the notion of *quotient* is represented by change of equivalence relation

Functions and operations

Given two "sets" (X, R, p) and (Y, S, q), an object $f : X \to Y$ is called an *operation*

If we have a proof of $R\ x_0\ x_1 \to S\ (f\ x_0)\ (f\ x_1)$ then f is called a *function* between (X, R, p) and (Y, S, q)

If f and g are two such functions, a natural equality of f and g is

$$(\Pi x : X) S\ (f\ x)\ (g\ x)$$

Functions and operations

The type $N : U$ has a natural equality relation, defined recursively

$$Eq\ 0\ 0 = N_1 \qquad Eq\ 0\ (S\ x) = Eq\ (S\ x)\ 0 = N_0 \qquad Eq\ (S\ x)\ (S\ y) = Eq\ x\ y$$

One can show that this equality is substitutive

$$(\Pi X : N \to U)(\Pi x_0\ x_1 : N)(Eq_N\ x_0\ x_1 \to P\ x_0 \to P\ x_1)$$

It follows from this that *any* object $f : N \to N$ is a function

$$(\Pi f : N \to N)(\Pi x_0\ x_1 : N)(Eq_N\ x_0\ x_1 \to Eq_N\ (f\ x_0)\ (f\ x_1))$$

Functions and operations

The situation is more subtle at type $F : (N \to N) \to N$

If F is a variable, it does not seem possible to show

$$(\Pi f\ g : N \to N)(Eq_{N\to N}\ f\ g \to Eq_N\ (F\ f)\ (F\ g))$$

On the other hand, one expects to be able to prove this implication for any *definable* F

This was formalized in a different context by R. Gandy in his interpretation of *extensional* type theory in *intensional* type theory

Functions and operations

This means that if we work with Bishop sets in type theory we need to keep explicitely the information that $F : (N \to N) \to N$ preserves the equality

However, for any definable terms of this type, this information can be inferred automatically

This does not seem satisfactory, neither conceptually nor in practice

Other question: what should be the equality on the type U?

Extensional versus Intensional

These problems motivated Martin-Löf to introduce a variation with an extensional equality (this did not solve the problem of what should be the equality at the type U)

Why this change? One reason was to get a system which is easier to use in practice

This is the system presented in Martin-Löf's book *Intuitionistic Type Theory*

Martin-Löf abandoned this approach in 1986 and went back to his 1972 version

Analysis of the Axiom of Choice

Given two Bishop sets X, R, p and Y, S, q, if we have a function $f : X \to Y$ such that

$$(\Pi y : Y)(\Sigma x : X)S\ (f\ x)\ y$$

then we have an *operation* $g : Y \to X$ such that

$$(\Pi y : Y)S\ (f\ (g\ y))\ y$$

but this operation g does not need to be a *function* in general

The *extensional* Axiom of Choice states that there exists a *function* $g : Y \to X$ satisfying

$$(\Pi y : Y)S\ (f\ (g\ y))\ y$$

Analysis of the Axiom of Choice

For instance there is a surjective function $\{-1, 0, 1\}^N \to [-1, 1]$ (where real numbers are represented as Cauchy sequences)

The extensional axiom of choice implies that this map has a section

This is not constructively valid, and implies LPO

One can show that the extensional axiom of choice implies EM!

One can also follow Zermelo's proof and show that extensional axiom of choice implies that any (Bishop) set can be well-ordered

Analysis of the Axiom of Choice

The extensional axiom of choice has no intuitive justification

The intensional axiom of choice is justified intuitively

According to Martin-Löf (2004), one should make a distinction between the two different usage of the Axiom of Choice

One argument used by Zermelo to defend the Axiom of Choice was that it was used implicitely by most mathematicians before (for instance the proof that a Cauchy sequences of real converges); however what was used was only the *intensional* form

A Table of Foundations

	ZFC	CZF	Topos Th.	MLTT	Univalent F.
EM	X				
ext AC	X				
DC	X	X		X	
extensional	X	X	X		X
impredicative	X		X		

A Table of Foundations

In this table DC is the axiom of *dependent choice*

$$\forall x.\exists y.R(x,y) \rightarrow \forall x.\exists f.f(0) = x \wedge \forall n.R(f(n), f(n+1))$$

The axiom of *countable choice* is

$$\forall n.\exists y.R(n,y) \rightarrow \exists f.\forall n.R(n, f(n))$$

Both are valid in MLTT and seemed important in Bishop's development of basic analysis

More recent works indicate that constructive results are often more elegant when stated and proved without these axioms

A Table of Foundations

One important aspect of MLTT is that it provides a framework in which one can express conceptual mathematics in a computational way

It can directly be seen as a functional programming language