# Type Theory and Constructive Mathematics

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# Univalent Foundations

Type = Proposition = Space

proof of equality = path

"All functions are continuous"

*New* connection between *logic* and *topology*

One previous connection was provided by topological models of intuitionism

Notice that Brouwer worked on both topology and intuitionism (and was one of the founder of combinatorial topology), as well as Poincaré who founded algebraic topology and was a precursor of intuitionism

Most of what I present has been discovered (and formalized in type theory) by Voevodsky

# Contractible types

In topology, a space $A$ is contractible iff there is a homotopy between the identity map of $A$ and a constant map

A space is (path) connected iff any two points are connected by a path

In type theory contr $A$ is defined to be

$$(\Sigma a : A)(\Pi x : A)\mathrm{Path}_A\ a\ x$$

This is *stronger* than being connected since the path connecting $a$ and $x$ has to be a *continuous function* of $x$

In topology $S^1$ is *not* contractible, but it is connected

# Contractible types

For any type $A$ and $a : A$ we can prove (elimination rule)

$$\text{contr } ((\Sigma x : A)\text{Path}_A \ a \ x)$$

This was the property of contractibility of the fibers of the path space used by Serre

In general

$$\text{contr } ((\Sigma x : A)B)$$

can be seen as a generalization of *unique existence* $\exists ! x : A.B$

Not only the witness satisfying $B(x)$ is uniquely determined but also the reason why it satisfies $B(x)$

# Contractible types

We see already at this point that in order to have a good correspondance with the fact that we have a homotopy between the constant map $\lambda x.a$ and the identity map $\lambda x.x$ we need to have the extensionality axiom

$$((\Pi x : A)\mathsf{Path}_A\ a\ x) \to \mathsf{Path}_{A\to A}\ (\lambda x.a)\ (\lambda x.x)$$

A lot of properties can be proved in MLTT with identity type without this axiom however

# Stratification of types

Define hlevel : $N \to U \to U$

$$\text{hlevel}\, 0\, A = \text{contr}\, A \qquad \text{hlevel}\, (S\ n)\, A = (\Pi x_0\, x_1 : A)\text{hlevel}\, n\, (\text{Path}_A\, x_0\, x_1)$$

and define prop = hlevel $(S\ 0)$ and set = hlevel $(S\ (S\ 0))$

For instance we have directly

$$(\neg\ A) \to \text{hlevel}\ (S\ n)\ A$$

We have prop $A \leftrightarrow (\Pi\ x_0\ x_1 : A)\text{Path}_A\ x_0\ x_1$

It follows that we have

$$\text{set}\ A \leftrightarrow \text{PI}_A$$

# Stratification of types

To be a *proposition* corresponds intuitively to being proof irrelevant which corresponds to what we understand in usual mathematics as proposition

If we have

$$(\Pi x : A)\mathrm{prop}\ B$$

we say that $B(x)$ is a *property* on $A$

In this case we can think of $(\Sigma x : A)B$ as a *subset* of $A$

The first projection $((\Sigma x : A)B) \to A$ is one-to-one

# Stratification of types

We can reformulate Hedberg's result as: any *discrete type* (type with a decidable equality) is a *(h)set*

$N$ is a set but not a proposition since $\neg(\mathrm{Path}_N\ 0\ 1)$ is inhabited

We have prop $N_0$ and contr $N_1$ and set $N_2$

Sets correspond intuitively to sets in mathematics (where it is irrelevant in what way we can prove the equality of two objects in this type)

# Stratification of types

**Theorem:** set $A \to ((\Pi x : A)\text{set } B) \to \text{set } (\Sigma x : A)B$

This was observed by Hedberg's and the motivation was actual formalization of domain theory in type theory (inverse limits of domain)

This notion of hset is important in representing mathematics in type theory

In the system SSReflect, one restricts oneself to *discrete* types, but one main use of this is because they satisfy Hedberg's Theorem, i.e. discrete types are sets

# Stratification of types

One can prove $(\Pi A : U)(\text{hlevel } n\ A \rightarrow \text{hlevel } (S\ n)\ A)$

For instance contr $A \rightarrow$ prop $A$

All this can be proved in MLTT extended with identity types

An "axiom" or a "property" of an object should be a type of hlevel 1

# Extensionality

Voevodsky formulates the extensionality axiom in the form

$$((\Pi x : A)\text{contr } B) \to \text{contr } (\Pi x : A)B$$

This implies, for $n : N$

$$((\Pi x : A)\text{hlevel } n\ B) \to \text{hlevel } n\ (\Pi x : A)B$$

In particular we have

$$((\Pi x : A)\text{prop } B) \to \text{prop } (\Pi x : A)B$$

# Extensionality

This implies

$$\text{prop (contr } A) \qquad \text{prop (set } A)$$

and more generally prop (hlevel $n$ $A$)

As noticed before, the correspondance with what happens in homotopy theory works well only if we add this axiom of extensionality

# Isomorphisms

Voevodsky found a way to state that a function is bijective as a *property*

If $f : A \to B$ and $b : B$ define Fiber $f\ b = (\Sigma x : A)\mathrm{Path}_B\ (f\ x)\ b$ and

$$\mathrm{IsWeq}\ f = (\Pi y : B)\mathrm{contr}\ (\mathrm{Fiber}\ f\ y)$$

We have prop (IsWeq $f$) so that IsWeq $f$ is a property of $f$

One can show

$$\mathrm{IsWeq}\ f$$

is *logically equivalent* to

$$(\Sigma g : B \to A)\mathrm{Path}_{A\to A}\ (\lambda x.x)\ (\lambda x.g\ (f\ x)) \times \mathrm{Path}_{B\to B}\ (\lambda y.y)\ (\lambda y.f\ (g\ y))$$

# Isomorphisms

We define $\mathrm{Weq}\ A\ B$ to mean $(\Sigma f : A \to B)\mathrm{IsWeq}\ f$

Clearly $\mathrm{Weq}\ A\ B \to (A \leftrightarrow B)$ however $\mathrm{Weq}\ A\ B$ is subtler than the logical equivalence of $A$ and $B$; it states that $A$ and $B$ are *isomorphic*

For instance we have

$$\mathrm{Weq}\ ((\Pi x : A)(\Sigma y : B)R\ x\ y)\ ((\Sigma f : A \to B)(\Pi x : A)R\ x\ (f\ x))$$

which is stronger than simply to state the axiom of choice

# Weak equivalence

Types of hlevel 3 correspond to *groupoids*

The notion of weak equivalence captures uniformely

isomorphisms of sets

(categorical) equivalence of groupoids

...

Equivalent groupoids are equal

# Isomorphisms

We have

$$\text{contr } A \leftrightarrow \text{Weq } N_1 \ A$$

and for instance, if $a : A$

$$\text{Weq } N_1 \ ((\Sigma x : A)\text{Path}_A \ a \ x)$$

One has also

$$\text{Weq } ((\Sigma x : A)(B + C)) \ ((\Sigma x : A)B + (\Sigma x : A)C)$$

Hence we have

$$\text{Weq } ((\Sigma x : A)(B_1 + \cdots + B_k)) \ ((\Sigma x : A)B_1 + \cdots + (\Sigma x : A)B_k)$$

# Isomorphisms

It follows that we have, if $a_1 \; \ldots \; a_k : A$

$$\mathsf{Weq}\ ((\Sigma x : A)(\mathsf{Path}_A\ a_1\ x + \cdots + \mathsf{Path}_A\ a_k\ x))\ N_k$$

If we define

$$x\epsilon[] = N_0 \qquad x\epsilon(y : ys) = \mathsf{Path}_A\ x\ y + x\epsilon ys$$

One can show Weq $((\Sigma x : A)x\epsilon xs)$ Fin $|xs|$ where

$$\mathsf{Fin}\ 0 = N_0 \qquad \mathsf{Fin}\ (S\ n) = N_1 + \mathsf{Fin}\ n$$

N.A. Danielsson has used this remark to give a nice definition of bag equality

$$Eq\ xs\ ys = (\Pi x : A)\mathsf{Weq}\ (x\epsilon xs)\ (x\epsilon ys)$$

# The Axiom of Univalence

We clearly have IsWeq $(\lambda x.x)$ and hence Weq $A$ $A$ for any type $A$

It follows that we have a map

$$\sigma : \text{Path}_U\ A\ B \to \text{Weq}\ A\ B$$

The *axiom of univalence* states that this map is a weak equivalence, hence we have

$$\text{Weq}\ (\text{Path}_U\ A\ B)\ (\text{Weq}\ A\ B)$$

Notice that the axiom of univalence, in the form stating that the map $\sigma$ is a weak equivalence, is itself a proposition, i.e. a type of hlevel 1

# Structural Identity Principle

Consider the structure of a type with a function and a constant

$$S = (\Sigma X : U)\ X \times (X \to X)$$

Define $A, a, f$ and $B, b, g$ to be isomorphic, i.e. we have $\sigma : A \to B$ and $\delta : B \to A$ such that

$$\text{Path}_{A \to A}\ (\delta \circ \sigma)\ (\lambda x.x) \qquad \text{Path}_{B \to B}\ (\sigma \circ \delta)\ (\lambda y.y)$$

$$\text{Path}_{B}\ (\sigma\ a)\ b \qquad \text{Path}_{A \to A}\ f\ (\delta \circ g \circ \sigma)$$

A consequence of the axiom of univalence is that if $A, a, f$ and $B, b, g$ are isomorphic then we have $\text{Path}_S\ (A, a, f)\ (B, b, g)$

# Structural Identity Principle

This can be stated as

Two mathematical structures that are *isomorphic* are *equal*

This principle corresponds to the usual practice of the mathematician, who wants to identify isomorphic structures

This principle is not satisfied by the set theoretic formalization, e.g. $A = \{0, 1\}$ and $B = \{1, 2\}$ are isomorphic but if we define $\varphi(X)$ to be the property $0 \in X$ we have $\varphi(A)$ and not $\varphi(B)$.

# Some open problem

Give a "meaning explanation" of the rules of equality (with extensionality and univalence principle)

Constructive model of the univalence: the Kan simplicial model uses classical logic in an essential way

(Voevodsky) If we build $\vdash t : N$ using the univalence axiom then there exists a numeral $k$ such that $\text{Path}_N\ t\ (S^k\ 0)$ can be proved (maybe using the univalence axiom)

We need a "propositional reflection": to each type $A$ we associate a *proposition* $A^*$ such that $A \to A^*$ and prop $P \to (A \to P) \to A^* \to P$ are provable. Constructive justification?

Quotient type