Kleene Algebra

Georg Struth

University of Sheffield

based on joint work with Jules Desharnais, Bernhard Möller and others

Motivation

program/system analysis requires formalisms that balance

- expressive interoperable modelling languages
- powerful proof procedures

modelling languages: e.g.

- relations used in Z or B
- functions/quantales used in refinement calculi
- modal logics/process algebras used for reactive/concurrent systems

proof procedures dominated by

- interactive proof checking
- model checking

Motivation

questions: are there formalisms that offer better balance

- unify/integrate relational, functional, modal reasoning?
- allow using off-the-shelf automated theorem provers (ATP systems)?

Motivation

questions: are there formalisms that offer better balance

- unify/integrate relational, functional, modal reasoning?
- allow using off-the-shelf automated theorem provers (ATP systems)?

answer: algebraic methods, in particular modal Kleene algebras (maybe)

benefits of algebraic approach:

- simple first-order equational calculi
- rich class of computationally meaningful models
- mechanisms for abstraction and (de)composition
- suitable for ATP systems

This Lecture Series

goal: introduce modal Kleene algebras as computational tools for modelling and analysing discrete dynamical systems

outline:

1. surveys foundations of (modal) Kleene algebras
2. discusses some computationally interesting models
3. sketches connection with popular computational logics
4. presents some (automation) examples

dual rôle of ATP: a new approach to

- computer mathematics: develop/analyse algebraic structures
- formal methods: develop/analyse programs and systems

apology: highly subjective and incomplete picture

Semirings, Actions and Propositions

semiring: $(S, +, \cdot, 0, 1)$ "ring without minus"

$$x + (y + z) = (x + y) + z \qquad x + y = y + x \qquad x + 0 = x$$

$$x(yz) = (xy)z \qquad x1 = x \qquad 1x = x$$

$$x(y + z) = xy + xz \qquad (x + y)z = xz + yz$$

$$x0 = 0 \qquad 0x = 0$$

interpretation: S represents actions of some discrete dynamical system

- $+$ models nondeterministic (angelic) choice (cf. next slide)
- $\cdot$ models sequential composition
- 0 models abortive action
- 1 models ineffective action

Semirings, Actions and Propositions

remarks:

- swapping multiplication yields opposite semiring
- semiring is idempotent if $x + x = x$
- idempotent semirings are naturally ordered by $x \leq y \Leftrightarrow x + y = y$
  hence $(S, +, 0)$ is upper semilattice with least element 0
- idempotency turns addition into choice

questions:

- how can the state space of the system be included?
- how can the “limit behaviour” of the system be described?

Semirings, Actions and Propositions

task: include the state space

test algebras: [ManesArbib] “Boolean centre”

- Boolean subalgebra $(\mathsf{test}(S), +, \cdot, \neg, 0, 1)$ embedded into $[0, 1]$ of S
- $+$ coincides with Boolean join
- $\cdot$ coincides with Boolean meet

remarks:

- Boolean algebra $\mathsf{test}(S)$ captures the main intuition behind state spaces
- elements of $\mathsf{test}(S)$ are sets of states
- alternative interpretations as propositions of a system or tests of a program

notation: $x, y, z \dots$ for actions; $p, q, r, \dots$ for tests/propositions

Kleene Algebras

task: describe “limit behaviour”

Kleene algebras: [Kozen] idempotent semiring with star satisfying

- unfold axiom $1 + xx^* \leq x^*$
- induction axiom $y + xz \leq z \Rightarrow x^*y \leq z$
- and their opposites

$$1 + x^*x \leq x^* \qquad y + zx \leq z \Rightarrow yx^* \leq z$$

Models of Kleene Algebra

Boolean semiring: structure $A_2 = (\{0,1\}, +, \cdot, {}^*, 0, 1)$ with operations

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$$0^* = 1^* = 1.$$

question: can you give the test algebra?

Models of Kleene Algebra

binary relation: set of ordered pairs on set M

$$R = \{(a,b) : a,b \in M\}$$

operations:

- empty relation: $\emptyset$ (the empty set)
- unit relation: $\Delta = \{(a,a) : a \in M\}$
- union: $R \cup S = \{(a,b) : (a,b) \in R \text{ or } (a,b) \in S\}$
- product: $R \circ S = \{(a,b) : (a,c) \in R \text{ and } (c,b) \in S \text{ for some } c \in M\}$
- star: $R^* = \bigcup_{i \geq 0} R^i$ where $R^0 = \Delta$ and $R^{i+1} = R \circ R^i$ for all $i \in \mathbb{N}$

remark: R^* corresponds to the reflexive transitive closure of R

Models of Kleene Algebra

fact: $(2^{M\times M}, \cup, \circ, ^*, \emptyset, \Delta)$ is a Kleene algebra, the full relation Kleene algebra over M

proof: check that relations satisfy Kleene algebra axioms. . .

fact: every subalgebra of a full relation Kleene algebra is again a Kleene algebra; a relation Kleene algebra

proof:

- logically, Kleene algebras are universal Horn theories
- a general theorem of universal algebra says that universal Horn theories are closed under subalgebras

Models of Kleene Algebra

question: can you define the test algebra of a relation Kleene algebra?

remarks:

- binary relations yield a standard semantics for (imperative) programs
- they model their input/output behaviour with respect to stores
- they capture nondeterminism and are very useful for specifications (even for functional programs)
- we will consider this semantics more abstractly below

Models of Kleene Algebra

question: the operations of Kleene algebras are precisely the regular operations; is there any connection with language theory?

words are finite sequences over a (finite) alphabet Σ

languages are sets of words

operations:

- empty language: $\emptyset$ (empty set)
- unit language: $\{\epsilon\}$ with empty word ϵ
- union: $L_1 \cup L_2$ as in set theory
- product: $L_1 \circ L_2 = \{w_1 w_2 : w_1 \in L_1 \text{ and } w_2 \in L_2\}$
- star: $L^* = \{w_1 w_2 \ldots w_n : w_i \in L \text{ and } n \geq 0\}$

Models of Kleene Algebra

fact: $(2^{\Sigma^*}, \cup, \circ, ^*, \emptyset, \{\epsilon\})$ is a Kleene algebra, the full language Kleene algebra over M

fact: every subalgebra of a full language Kleene algebra is again a Kleene algebra; a language Kleene algebra

regular subsets/events: obtained from finite subsets of Σ^* by finite number of regular operations

consequence: strong link between Kleene algebras and regular languages

Models of Kleene Algebra

slogan: Kleene algebras are algebras of regular events

- Kozen has shown that an equation holds in Kleene algebras iff it holds about regular events/expressions
- mathematically, the algebra of regular events over Σ is the free Kleene algebra generated by Σ

consequence: equations in Kleene algebras can be decided by automata

remarks:

- this correspondence motivates the name “Kleene algebra”
- universal Horn theory of Kleene algebras is undecidable (Post)
- there is no finite equational axiomatisation for the equational theory of regular events

Models of Kleene Algebra

paths: finite sequences of states from P; empty path ϵ

path product: glue paths together

$$\sigma.p \cdot p.\sigma' = \sigma.p.\sigma' \qquad \sigma.p \cdot q.\sigma' \quad \text{undefined}$$

operations on sets of paths:

- $P_1 \circ P_2 = \{\pi_1 \cdot \pi_2 : \pi_1 \in P_1, \pi_2 \in P_2 \text{ and } \pi_1 \cdot \pi_2 \text{ defined}\}$
- other operations as usual (what is multiplicative unit?)

consequence: sets of paths form path Kleene algebras

Models of Kleene Algebra

trace: alternating sequence $p_0 a_0 p_1 a_1 p_2 \dots p_{n-2} a_{n-1} p_{n-1}$, $p_i \in P$, $a_i \in A$

trace product: $\sigma.p \cdot p.\sigma' = \sigma.p.\sigma'$ $\sigma.p \cdot q.\sigma'$ undefined

operations on sets of traces:

- $T_1 \circ T_2 = \{\tau_1 \cdot \tau_2 : \tau_1 \in T_1, \tau_2 \in T_2$ and $\tau_1 \cdot \tau_2$ defined$\}$
- other operations as usual (what is multiplicative unit?)

consequence: sets of traces form trace Kleene algebras

Relationship Between Models

special cases: essentially by forgetting structure in trace MKA

- path/language Kleene algebras forget actions/propositions
- relation Kleene algebras forget sequences between endpoints

property: (equational) properties are inherited by (relations), paths, languages

remark:

- traces, paths, languages, relations are computationally interesting models
- Kleene algebras are applicable in interoperable contexts

Further Models

matrix model: consider $n \times n$ matrices over Kleene algebra

- 0 and 1 are zero and unit matrix
- $+$ and $\cdot$ are standard matrix addition and multiplication
- star defined by partitioning a non-singleton matrix into submatrices a, b, c, d, with a and d square, and setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} f^* & f^*bd^* \\ d^*cf^* & d^* + d^*cf^*bd^* \end{pmatrix}$$

where $f = a + bd^*c$

fact: matrices over Kleene algebras form Kleene algebras

Digression: Automata, Algebraically

finite automaton: [Conway] (u, A, v) with

- u 0-1 vector of start states
- v 0-1 vector of accepting states
- A transition matrix over Kleene algebra

language accepted by automaton is element $u^T A^* v$ of Kleene algebra

simple automaton: transition matrix of form

$$A = J + \sum_{a \in \Sigma} a \cdot A_a$$

for 0-1 matrices J (ϵ-matrix) and A_a

fact: automata theory can be developed from this angle

Digression: Automata, Algebraically

example: consider automaton with states $\{p, q\}$, alphabet $\{a, b\}$, start state p, accept state q, and transitions

$$p \rightarrow_a p \qquad q \rightarrow_a q \qquad p \rightarrow_b q \qquad q \rightarrow_b q$$

algebraic automaton:

$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a+b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

language accepted:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a+b \end{pmatrix}^* \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a^* & a^*b(a+b)^* \\ 0 & (a+b)^* \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= a^*b(a+b)^* \end{aligned}$$

Further Models

tropical semiring: $(N_\infty, \min, +, \infty, 0, {}^*)$ is Kleene algebra if $n^* = 0$ for all $n \in N_\infty$

applications:

- combinatorial optimisation
- path problems (encoded via matrices)
- rich mathematical theory

remark: this area alone would deserve a lecture series. . .

remark: max-plus semiring on $N_{-\infty}$ cannot be extended to a Kleene algebra

Kleene Algebras and Regular Programs

fact: KAs capture while-programs/guarded commands in various semantics

$$\begin{aligned} \textsf{abort} &= 0 \\ \textsf{skip} &= 1 \\ x;y &= xy \\ \textsf{if } p \textsf{ then } x \textsf{ else } y &= px + \neg p y \\ \textsf{while } p \textsf{ do } x &= (px)^* \neg p \end{aligned}$$

remarks:

- the usual semantic mappings have been suppressed
- the assignment rule cannot be modelled in this setting
- it can be modelled in an extension formalising substitution

Calculus of Kleene Algebras

rich calculus: all regular identities hold in Kleene algebras. e.g.,

$$1 \leq x^* \quad x \leq x^* \quad xx^* \leq x^* \quad x^*x \leq x^* \quad 1 + xx^* = x^* \quad 1 + x^*x = x^*$$

$$x^*x^* = x^* \quad x^{**} = x^* \quad (xy)^*x = x(yx)^* \quad (x+y)^* = x^*(yx^*)^*$$

some quasi-identities:

$$x \leq y \Rightarrow xz \leq yz \qquad x \leq y \Rightarrow zx \leq zy$$

$$x \leq y \Rightarrow x + z \leq y + z \qquad x \leq y \Rightarrow x^* \leq y^*$$

$$x \leq 1 \Rightarrow x^* = 1 \qquad x \leq y \Rightarrow x^* \leq y^*$$

$$xz \leq zy \Rightarrow x^*z \leq zy^* \qquad zx \leq yz \Rightarrow zx^* \leq y^*z$$

$$xy \leq y \Rightarrow x^*y \leq y \qquad yx \leq y \Rightarrow yx^* \leq y$$

more results: `www.dcs.shef.ac.uk/∼georg/ka`

Example: Church-Rosser Theorem and Concurrency Control

abstract reduction: rewrite relations as binary relations

Church-Rosser theorem: $y^*x^* \leq x^*y^* \Rightarrow (x+y)^* \leq x^*y^*$

proof:

- $(x+y)^* = (y^*x^*)^*$ is regular identity
- it suffices to show $y^*x^* \leq x^*y^* \Rightarrow (y^*x^*)^* \leq x^*y^*$ (induction over number of peaks)
- by star induction it suffices to show $1 + y^*x^*x^*y^* \leq x^*y^*$
- this splits into $1 \leq x^*y^*$ and $y^*x^*x^*y^* \leq x^*y^*$
- the first identity (base case) is trivial
- for the second one (induction step) we calculate

$$y^*x^*x^*y^* = y^*x^*y^* \leq x^*y^*y^* = x^*y^*$$

Example: Church-Rosser Theorem and Concurrency Control

discussion:

- induction on number of peaks without external induction measure
- in concurrency control $(x + y)^*$ corresponds to nondeterministic loop
- this loop can be separated if y^*x^* sequences can be rearranged
- theorem holds also in trace, path and language model

outlook:

- abstract part of Church-Rosser theorem in λ-calculus can be proved in a similar way
- many other rewrite theorems can be proved as well

further application: transformation of while programs

General Remarks on Kleene Algebras

conclusion: Kleene algebras

- focus on the essential operations for modelling programs and discrete systems
- support abstract and concise reasoning within first-order logic
- have rich class of computationally meaningful models
- are strongly linked with decision procedures
- can be integrated with ATP systems (later. . .)

remark: induction axiom $y + xz \leq z \Rightarrow x^*y \leq z$ and dual

- provide star elimination rules
- support some inductive reasoning
- we will see further examples later

General Remarks on Kleene Algebras

variations: (see below) by weakening some axioms

- demonic refinement algebras for reasoning about total program correctness in predicate transformer models
- probabilistic Kleene algebras for analysing probabilistic protocols via probability transformers
- game algebras that capture combined angelic and demonic behaviour of agents via gameboard semantics
- basic process algebras

limitations:

- terminating and diverging behaviour cannot be expressed
- "nonregular" induction is not possible
- reasoning about concrete applications is model-sensitive

Adding Modalities

motivation:

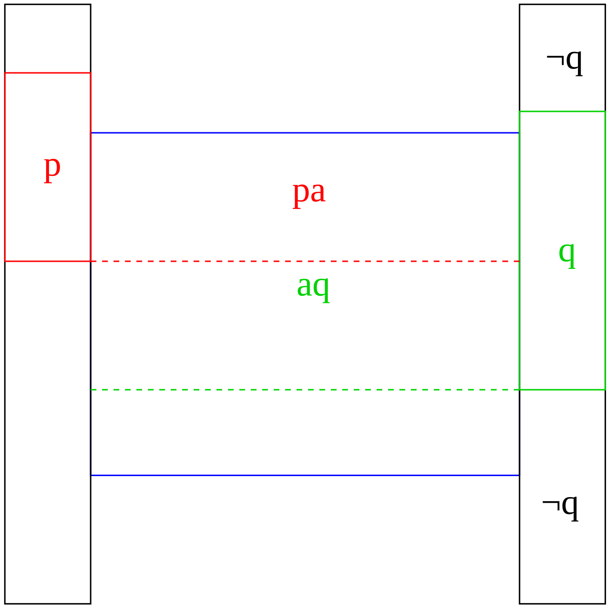
- many applications require different approach to actions/propositions
- systems dynamics is often modelled via state transitions;
  i.e. mappings from states to states
- various logics "use" Kleene algebras, but what is the precise connection?

idea: modal approach

- actions/propositions via Kripke frames
- system dynamics via images/preimages $\langle x|p \,/\, |x\rangle p$
- preimages via axiomatisation of domain
- images via axiomatisation of codomain

State Transitions

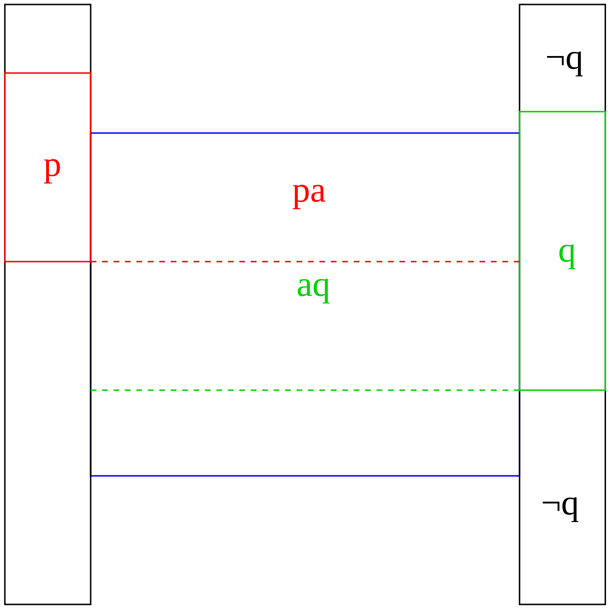
express: "terminating program a starting from store p creates store q"



in idempotent semiring: $pa \leq aq$ or equivalently $pa\neg q = 0$

State Transitions

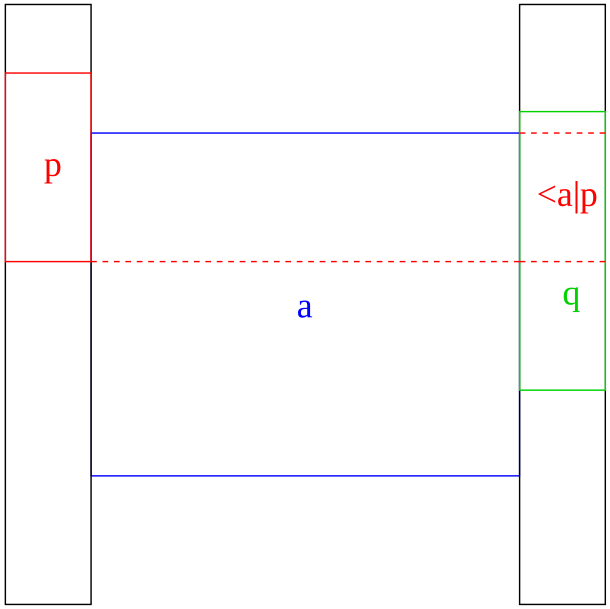
proof of equivalence



$$pa = pa(q+\neg q) = paq + pa\neg q = paq + 0 \leq aq \qquad pa\neg q \leq aq\neg q = a0 = 0$$

State Transitions

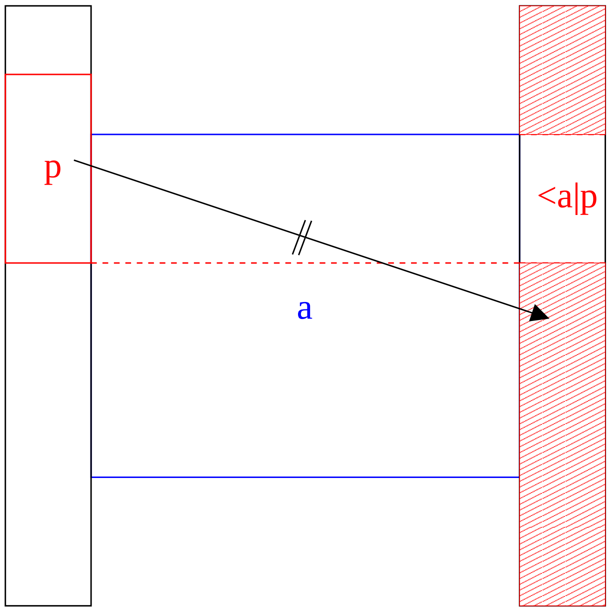
alternative: "q contains a-image of p"



question: how can we model images/preimages directly in idempotent semirings?

Image

relational model: complement of image of set p under relation a



is greatest set that does not admit an a-transition from p

Domain on Trace, Path, Language and Relation Semirings

intuition:

- relation semiring: $d(R) = \{a : (a, b) \in R\}$
- trace semiring: $d(T) = \{p : p = \text{first}(\tau) \text{ and } \tau \in T\}$
- path semiring: analogous
- language semiring: $d(\emptyset) = \emptyset$ and $d(L) = \{\epsilon\}$ else

general idea:

- domain as mapping $d : S \to S$ on semiring S
- $d(x)$ models states at which action x is enabled
- $d(x)$ should be
  - ≤ 1
  - least left preserver of x: $x \leq px \Leftrightarrow d(x) \leq p$
- equational axioms would be nice

Domain Semirings

general idea:

- axiomatise domain as mapping $d : S \to S$ on semiring S
- $d(x)$ models states at which action x is enabled
- $d(x)$ should be
  - ≤ 1
  - least left preserver of x: $x \leq px \Leftrightarrow d(x) \leq p$
    where px models restriction of action x to states in p
- equational axioms would be nice

question: what would be the type of p?

Domain Semirings

domain semiring: semiring with mapping $d : S \to S$ that satisfies

$$x + d(x)x = d(x)x \qquad d(xy) = d(xd(y)) \qquad d(x + y) = d(x) + d(y)$$

$$d(x) + 1 = 1 \qquad d(0) = 0$$

some intuition:

- axiom 1: $x \leq d(x)x$ means that domain is a left preserver
- axiom 2: $d(xy)$ is local on y through its domain
- axiom 3: enabling a choice means enabling one alternative or the other
- axiom 4: domain is smaller than 1 (cf. next slide)
- axiom 5: the abortive action is never enabled

Domain Semirings

property: every domain semiring is automatically idempotent

further properties: the axioms

- are irredundant (use model generator Mace4)
- cannot be weakened to inequalities (Mace4)
- imply least left preservation
- imply many "natural properties" (cf. next slides)

domain elements: $d(x) = x$ says "x is domain element"

Properties of Domain

fact: Let S be a domain semiring. Let $x, y \in S$ and let $p \in d(S)$. Then

- $d(x)x = x$ (domain is a left invariant)
- $d(p) = p$ (domain is a projection)
- $d(xy) \leq d(x)$ (domain increases for prefixes)
- $x \leq 1 \Rightarrow x \leq d(x)$ (domain expands subidentities)
- $d(x) = 0 \Leftrightarrow x = 0$ (domain is very strict)
- $d(1) = 1$ (domain is co-strict)
- $x \leq y \Rightarrow d(x) \leq d(y)$ (domain is isotone)
- $d(px) = pd(x)$ (domain elements can be exported)
- $d(x)d(x) = d(x)$ (domain elements are multiplicatively idempotent)
- $d(x)d(y) = d(y)d(x)$ (domain elements commute)
- $x \leq px \Leftrightarrow d(x) \leq p$ (domain elements are least left-preservers)
- $xy = 0 \Leftrightarrow xd(y) = 0$ (domain is weakly local)

Domain Algebra

question: how can we relate domain elements with tests?

property: for every domain semiring S, the sub-structure $(d(S), +, \cdot, 0, 1)$ is a bounded distributive lattice

proof: (with ATP)

1. check closure properties, $d(1) = 1$ and $d(0) = 0$
2. this gives sub-semiring
3. $d(x) \leq 1$ is axiom and $d(x)d(x) = d(x)$
4. but semirings satisfying these two properties are distributive lattices [Birkhoff]

notation:

- $(d(S), +, \cdot, 0, 1)$ is called domain algebra of S
- $p, q, r \ldots$ for domain elements

Domain Algebra

question: how can we enrich the domain algebra?

answer: (examples)

1. Heyting algebra: add Galois connection (and closure condition for $\rightarrow$)

$$pq \leq r \Leftrightarrow p \leq q \rightarrow r$$

2. Boolean algebra: add antidomain operation $a : S \rightarrow S$ with axioms

$$d(x) + a(x) = 1 \qquad d(x)a(x) = 0$$

Boolean Domain Algebra

assume: semiring that satisfies the domain/antidomain axioms

consequence: $d(S)$ is the largest Boolean subalgebra of S, so

$$d(S) = \mathsf{test}(S)$$

properties: (ATP)

- $a^2(x) = d(x)$
- $a(x)$ is greatest left annihilator of x: $\quad px = 0 \Leftrightarrow p \leq a(x)$

consequence:

- d can be replaced by a^2
- many domain/antidomain axioms become redundant
- axiomatisation can be simplified
- this yields. . .

Boolean Domain Semirings

Boolean domain semiring: semiring S with mapping $a : S \to S$ that satisfies

$$a(x)x = 0 \qquad a(xy) \leq a(xa^2(y)) \qquad a^2(x) + a(x) = 1$$

remarks:

- ATP/model search is very helpful in this development
- simple axioms induce rich modal calculus. . .

Modal Semirings

idea: define forward/backward diamonds as preimages/images

$$|x\rangle p = d(xp) \qquad \langle x|p = d^{\circ}(px)$$

where codomain operation d° is dual of domain

consequence:

- we have $|x\rangle 0 = 0$ and $|x\rangle(p+q) = |x\rangle p + |x\rangle q$
- this yields
  - distributive lattices with operators
  - Heyting algebras with operators
  - Boolean algebras with operators

convention: we will call KAs with Boolean domain modal KAs (MKAs)

Modalities, Symmetries, Dualities for Boolean Domain

demodalisation: $|x\rangle p \le q \Leftrightarrow \neg q x p \le 0$ $\quad$ $\langle x|p \le q \Leftrightarrow p x \neg q \le 0$

dualities:

- de Morgan: $|x]p = \neg|x\rangle\neg p$ $\quad$ $[x|p = \neg\langle x|\neg p$
- opposition: $\langle x|,\ [x| \Leftrightarrow |x\rangle,\ |x]$

symmetries:

- conjugation: $(|x\rangle p)q = 0 \Leftrightarrow p(\langle x|q) = 0$
- Galois connection: $|x\rangle p \le q \Leftrightarrow p \le [x|q$

benefits: rich calculus (automatically verified)

- symmetries as theorem generators
- dualities as theorem transformers

Kleene Modules

Kleene module: [Leiß06] structure $(K, L, :)$ with

$$(x+y)p = xp + yp \qquad x(p+q) = xp + xq \qquad (xy)p = x(yp)$$

$$1p = p \qquad x0 = 0 \qquad xp + q \leq p \Rightarrow x^*q \leq p$$

remark: scalar product $:$ omitted

fact: modal Kleene algebras are Kleene modules with $: = \lambda x \lambda p. |x\rangle p$

consequence: close relationship with computational logics

MKAs **and Propositional Dynamic Logic**

fact: MKAs are dynamic/test algebras

proof:

- dynamic algebras are almost Kleene modules
- main task is to show equivalence of
  - module induction law $|x\rangle p + q \leq p \Rightarrow |x^*\rangle q \leq p$
  - Segerberg axiom $|x^*\rangle p - p \leq |x^*\rangle(|x\rangle p - p)$

extensionality: $(\forall p. |x\rangle p = |y\rangle p) \Rightarrow x = y$

intuition: extensionality forces Kripke-style models

corollary: extensional MKAs are essentially propositional dynamic logics

MKAs **and Propositional Dynamic Logic**

benefits: MKA offers

- simpler/more modular axioms
- richer model class (beyond Kripke frames)
- more flexible setting

perspective:

- simple automated reasoning about programs and systems with off-the-shelf ATP systems
- easily extendable to the automation of first-order variants, e.g.,

$$\exists x \forall p \exists q.(|x\rangle f(p) \leq |x\rangle g(q) \rightarrow |x]h(p,q) = 0)$$

- some temporal logics and Hoare logics subsumed

MKAs and Linear Temporal Logic

encoding:

- temporal operators (use one single action x)

$$Xp = |x\rangle p \qquad Fp = |x^*\rangle p \qquad Gp = |x^*]p \qquad pUq = |(px)^*\rangle q$$

- initial state $\mathsf{init}_{\mathsf{x}} = [x|0$ “there’s nothing before the beginning”
- validity of temporal implications $\sigma \models p \to q \Leftrightarrow \mathsf{init}_{\mathsf{x}} \cdot p = q$
- tests now model sets of traces and x models the abstract tail relation

MKAs **and Linear Temporal Logic**

LTL axioms: von Karger's variant of [Manna/Pnueli]

$$\begin{aligned}
|(px)^*\rangle q = q + p|x\rangle|(px)^*\rangle q &\qquad \langle(xp)^*|q = q + p\langle(xp)^*|\langle x|q \\
|(px)^*\rangle 0 \le 0 &\qquad \langle x|0 = 1 \\
|x^*](p \to q) \le |x^*]p \to |x^*]q &\qquad [x^*|(p \to q) \le [x^*|p \to [x^*|q \\
|x^*]p \le p|x]|x^*]p &\qquad |x^*](p \to |x]p) \le |x^*](p \to |x^*]p) \\
p \le [x||x\rangle p &\qquad p \le |x]\langle x|p \\
\mathsf{init}_x \le |x^*](p \to [x|q) \to |x^*](p \to [x^*|q) &\qquad \mathsf{init}_x \le |x^*]p \to |x^*][x|p \\
|x](p \to q) = |x]p \to |x]q &\qquad [x|(p \to q) = [x|p \to [x|q \\
\langle x|p \le [x|p &\qquad |x\rangle p = |x]p
\end{aligned}$$

MKAs **and Linear Temporal Logic**

fact:

1. blue axioms are theorems of MKA
2. violet axioms express linearity of models (in MKA)

benefits:

- reasoning about infinite-state systems possible
- first-order temporal reasoning
- trace model available

remark:

- CTL also subsumed
- CTL* needs additional fixed points (and quantale-based setting)

MKAs and Hoare Logic

fact: MKA subsumes (propositional) Hoare logic

explanation: this is Hoare logic without the assignment rule

convention: Kleenean notation for syntax and semantics

validity of Hoare triple: $\models \{p\}\ x\ \{q\} \Leftrightarrow \langle x|p \leq q$

“terminating program x starting from store p creates store q”

validity of implication: $\models p \to q \Leftrightarrow p \leq q$

example: validity of while rule $\vdash_{\mathsf{MKA}} \langle x|pq \leq q \Rightarrow \langle (px)^{*}\neg p|q \leq \neg pq$

MKAs and Hoare Logic

benefits:

- weakest liberal precondition semantics for free in MKA ($\mathsf{wlp}(x, p) = |x]p$)
- soundness and completeness of Hoare logic are easy in MKA
- formalism of Hoare logic is dissolved in modal setting
- relative completeness not an issue. . .

Propositional Hoare Logics

Hoare calculus: inference rules

- abort: $\models \{p\}\ \textsf{abort}\ \{q\}$
- skip: $\models \{p\}\ \textsf{skip}\ \{p\}$
- assignment: $\models \{q[e/x]\}\ x := e\ \{q\}$
- composition: $\models \{p\}\ x\ \{q\}, \{q\}\ y\ \{r\} \Rightarrow \{p\}\ x\,;\,y\ \{r\}$
- conditional: $\models \{p \wedge q\}\ x\ \{r\}, \{\neg p \wedge q\}\ y\ \{r\} \Rightarrow \{q\}\ \textsf{if}\,p\,\textsf{then}\,x\,\textsf{else}\,y\ \{r\}$
- while: $\models \{p \wedge q\}\ x\ \{q\} \Rightarrow \{q\}\ \textsf{while}\,p\,\textsf{do}\,x\ \{\neg p \wedge q\}$
- weakening: $\models p_1 \to p, \{p\}\ x\ \{q\}, q \to q_1 \Rightarrow \{p_1\}\ x\ \{q_1\}$

Soundness

Hoare calculus: coding validity in MKA

- abort: $\langle 0|p \le p$
- skip: $\langle 1|p \le p$
- assignment: expressiveness assumption
- composition: $\langle x|p \le q, \langle y|q \le r \Rightarrow \langle xy|p \le r$
- conditional: $\langle x|(pq) \le r, \langle y|(\neg pq) \le r \Rightarrow \langle px + \neg py|q \le r$
- while: $\langle x|(pq) \le q \Rightarrow \langle (px)^* \neg p|q \le \neg pq$
- weakening: $p_1 \le p, \langle x|p \le q, q \le q_1 \Rightarrow \langle x|p_1 \le q_1$

Soundness

Hoare calculus: coding validity in operator Kleene algebra

- abort: $0 \leq f$
- skip: $1 \leq 1$
- assignment: expressiveness assumption
- composition: $\langle xy| \leq \langle y|\langle x|$
- conditional: $\langle px + \neg py| \leq \langle x|\langle p| + \langle y|\langle \neg p|$
- while: $\langle x|\langle p|f \leq f \Rightarrow \langle (px)^* \neg p|f \leq \langle \neg p|f$
- weakening: $f_1 \leq f, hf \leq g, g \leq g_1 \Rightarrow hf_1 \leq g_1$

Soundness

Hoare calculus: inference rules are theorems in operator Kleene algebra

- abort: $0 \le f$ trivial
- skip: $1 \le 1$ trivial
- assignment: expressiveness assumption
- composition: $\langle xy| \le \langle y|\langle x|$ contravariance
- conditional: $\langle px + \neg py| \le \langle x|\langle p| + \langle y|\langle \neg p|$ decomp., contravar.
- while: $\langle x|\langle p|f \le f \Rightarrow \langle (px)^* \neg p|f \le \langle \neg p|f$ next slide. . .
- weakening: $f_1 \le f, hf \le g, g \le g_1 \Rightarrow hf_1 \le g_1$ isotonicity

Soundness

proof of while-rule $\langle x|\langle p|f \leq f \Rightarrow \langle (px)^{*}\neg p|f \leq \langle \neg p|f$

$$
\begin{aligned}
\langle x|\langle p|f \leq f &\Leftrightarrow \langle px|f \leq f && (\text{ contravariance }) \\
&\Rightarrow \langle (px)^{*}|f \leq f && (\text{ induction }) \\
&\Rightarrow \langle \neg p|\langle (px)^{*}|f \leq \langle \neg p|f && (\text{ isotonicity }) \\
&\Leftrightarrow \langle (px^{*})\neg p|f \leq \langle \neg p|f && (\text{ contravariance })
\end{aligned}
$$

proposition: propositional Hoare logic is sound wrt algebraic semantics

Decidability

Hoare formulas: quasi-identities in modal Kleene algebra

$$\langle x_1|p_1 \le q_1, \dots, \langle x_n|p_n \le q_n \Rightarrow \langle a_0|p_0 \le q_0$$

decision procedure: (PSPACE)

1. demodalisation: rewrite as equivalent quasi-identity in Kleene algebra

$$p_1x_1\neg q_1 \le 0, \dots, p_nx_n\neg q_n \le 0 \Rightarrow p_0x_0\neg q_0 \le 0$$

2. hypothesis elimination: reduce to equivalent identity $s' \le t'$
3. apply PSPACE decision procedure for equational theory

MKAs and Hoare Logic

perspective:

- full automation of Hoare logic seems possible
- assignment rule requires formalising substitution
- handling numbers or data types is so far difficult for ATP systems
- approach extends to total correctness

Divergence and Termination

∇-Kleene module: Kleene module $(K, L, :)$ with divergence $^\nabla : K \to L$ satisfying

- ∇-unfold $x^\nabla \leq xx^\nabla$
- ∇-coinduction $p \leq xp + q \Rightarrow p \leq x^\nabla + x^* q$

remark: scalar product symbol omitted

interpretation:

1. for modal Kleene algebra, x^∇ denotes those states from which infinite behaviour may start
2. if K models finite actions and L infinite actions, then x^∇ is the infinite iteration of finite action x

Divergence and Termination

fact: if L is Boolean algebra, then ∇-coinduction is equivalent to

$$p \leq xp \Rightarrow p \leq x^{\nabla}$$

final part: $\mathsf{max}_x(p) = p - xp$ models final part of p w.r.t. x

termination: action x terminates if $x^{\nabla} = 0$

property: if L is Boolean algebra, then x terminates iff

$$\mathsf{max}_x(p) = 0 \Rightarrow p = 0$$

remark: this captures set-theoretic notion of Noethericity

Divergence and Termination

trace model:

- let K be a trace Kleene algebra
- let L be a set of infinite traces under union
- define, for $\tau \in K$ and $\pi \in L$ the scalar product $\tau : \pi$ like product of finite traces
- lift that product to sets of traces
- define $x^{\nabla} = \{\pi \in L : \pi = \tau_0 \cdot \tau_1 \cdot \ldots \text{ with } \tau_i \in K \text{ for } i \geq 0\}$

Then $(K, L, :, {}^{\nabla})$ is a (full trace) ∇-Kleene module

special cases: path and language ∇-Kleene modules

consequence: ∇-Kleene modules useful for integrated finite/infinite behaviour

Divergence and Termination

fact: divergence and termination can be equationally axiomatised

- $p \leq x^{\nabla} + x^{*}\mathsf{max}_x(p)$ is equivalent to ∇-coinduction
- $p \leq x^{*}\mathsf{max}_x(p)$ is equivalent to termination

remark: L must be Boolean algebra

intuition: p either leads to divergence or to final states after a finite iteration

perspective:

- characterisation dual to Segerberg's axiom
- equational approach to finite and infinite behaviours of discrete dynamical systems
- very suitable for ATP systems (see below)

Domain on Sub-Semirings

near-semiring: structure $(S, +, \cdot)$ such that

- $(S, +)$ and $(S, \cdot)$ are semigroups
- right distributivity law $(x + y)z = xz + yz$ holds

pre-semiring: left pre-isotone near-semiring $x + y = y \Rightarrow zx + zy = zy$

units: 0, 1 or

- deadlock $x + \delta = x \quad \delta x = \delta$.
- silent action $x\tau = x$

Domain on Sub-Semirings

basic process algebra: idempotent near-semiring $(S,+,\cdot,{}^*)$ or $(S,+,\cdot,{}^*,\delta,\tau)$

game algebra: idempotent pre-semiring $(S,+,\cdot,0,1)$

probabilistic Kleene algebra: idempotent pre-semiring $(S,+,\cdot,{}^*,0,1)$

demonic refinement algebra: idempotent semiring $(S,+,\cdot,{}^*,{}^\infty,\delta,1)$

Domain on Sub-Semirings

	NS_δ^τ	NS_δ^1	PS_δ^1
$a(x)x = \delta$		$\surd$	$\surd$
$a(xy) \leq a(xa^2(y))$		$\surd$	$\surd$
$a^2(x) + a(x) = 1$	$\surd$	$\surd$	$\surd$
$a(x+y) = a(x)a(y)$		$\surd$	
$x = d(x)x$	$\surd$		
$d(xy) = d(xd(y))$	$\surd$		
$d(x+y) = d(x) + d(y)$	$\surd$		
$d(\delta) = \delta$	$\surd$		
$d(x)d(y) = d(y)d(x)$	$\surd$		
$d(a(x)) = a(x)$	$\surd$		

NS: near-semiring, PS: pre-semiring

Domain on Sub-Semirings

conclusion:

- domain can still be defined on sub-semirings
- this models enabledness conditions for games, processes and actions in protocols
- semiring domain axioms suffice for probabilistic Kleene algebras and demonic refinement algebras
- domain does not induce modal operators

Automation Examples

observation: ATP systems have rather been neglected in formal methods

idea: combine MKAs with ATPs and counter example generators

results: experiments with various ATPs (Prover9, SPASS, Waldmeister,. . .)

- ~ 500 theorems automatically proved
- successful case studies in program refinement, termination, . . . analysis

benefits:

- special-purpose calculi made redundant
- generic flexible library of lemmas
- new style of verification

Automating Bachmair and Dershowitz's Termination Theorem

theorem: [BachmairDershowitz86] *termination of the union of two rewrite systems can be separated into termination of the individual systems if one rewrite system quasicommutes over the other*

formalisation: ∇-Kleene module over semilattice

encoding:

- quasicommutation $yx \leq x(x+y)^*$
- separation of termination $(x+y)^\nabla = 0 \Leftrightarrow x^\nabla + y^\nabla = 0$

statement: termination of x and y can be separated if x quasicommutes over y

remark: posed as challenge by Ernie Cohen in 2001

Automating Bachmair and Dershowitz's Termination Theorem

results: SPASS finds an extremely short proof in < 5min

$$
\begin{aligned}
(x+y)^\nabla &= y^\nabla + y^* x (x+y)^\nabla && \text{(sum unfold)}\\
&\le y^\nabla + x(x+y)^*(x+y)^\nabla && \text{(strong quasicommmutation)}\\
&= y^\nabla + x(x+y)^\nabla && \text{(since } z^\omega = z^* z^\omega\text{)}\\
&\le x^\nabla + x^* y^\nabla && \text{(coinduction)}\\
&= 0 && \text{(assumption } x^\nabla = y^\nabla = 0\text{)}
\end{aligned}
$$

Automating Bachmair and Dershowitz's Termination Theorem

surprise: proof reveals new refinement law

$$yx \leq x(x+y)^* \Rightarrow (x+y)^\nabla = x^\nabla + x^* y^\nabla$$

for separating infinite loops

remarks:

- reasoning essentially coinductive
- theorem holds in large class of models
- translation safe since relations form ∇-Kleene modules

Automating the DBW-Theorem

lazy commutation: $yx \leq x(x+y)^* + y$

theorem: [Doornbos/Backhouse/van der Woude]
if x lazily commutes over y then termination of x and y can be separated

comment: this generalisation is much more difficult

lemma: x lazily commutes over y iff

$$yx^* \leq x(x+y)^* + y$$

proof: $44.23s$ by Prover9.

Automating the DBW-Theorem

proof: (non-trivial direction of DBW-theorem)

1. abbreviate $\nabla = (x+y)^{\nabla}$
2. assume that x and y terminate
3. for $\nabla = 0$ it suffices to show $\mathsf{max}_y(\mathsf{max}_x(\nabla)) = 0$
4. this is equivalent to $\mathsf{max}_x(\nabla) \leq y\mathsf{max}_x(\nabla)$
5. we calculate

$$\begin{aligned}\nabla &= x\nabla + y\nabla \leq x\nabla + yx^{*}\mathsf{max}_x(\nabla) \leq x\nabla + x(x+y)^{*}\mathsf{max}_x(\nabla) + y\mathsf{max}_x(\nabla)\\ &\leq x\nabla + x(x+y)^{*}\nabla + y\mathsf{max}_x(\nabla) = x\nabla + y\mathsf{max}_x(\nabla)\end{aligned}$$

6. the claim now follows from the Galois connection for complementation and the definition of max_x

remark: the second step uses the equational characterisation of termination

Automating the DBW-Theorem

remarks:

- proof is much more compact than previous approaches
- for the first time in first-order setting
- theorem holds again in large model class
- main calculation could again be automated
- full automation remains a challenge

Automating a Modal Correspondence Result

modal logic: Löb's formula $\Box(\Box p \to p) \to \Box p$

translation to MKA/Kleene modules: $xp \leq x(p - xp) = x\mathsf{max}_x(p)$

intuition: all states with transitions into p are states from which no further transitions are possible

remark: this would correspond to Noethericity if x is transitive ($xx \leq x$)

reminder: two equivalent characterisations of Noethericity

- $p \leq x^*\mathsf{max}_x(p)$ (x pre-Löbian)
- $\mathsf{max}_x(p) = 0 \Rightarrow p = 0$ (x Noetherian)

Automating a Modal Correspondence Result

property: for every x in some ∇-Kleene module

(i) x Löbian $\Rightarrow$ x Noetherian
(ii) x Noetherian $\Leftrightarrow$ x pre-Löbian (see above)
(iii) x pre-Löbian and $x = xx$ $\Rightarrow$ x Löbian

proofs: with Prover9 in ∇-Kleene algebra

(i) $\leq 4s$
(ii) $\leq 4s$ and $\leq 20s$ (hypothesis learning)
(iii) $\leq 1s$ (hypothesis learning)

remark: this is a modal correspondence result

- Noethericity corresponds to frame property
- proof is calculational and automated
- model theory is normally used

Automating Hoare Logic

algorithm: integer division n/m

```
fun DIV = k:=0;l:=n;
          while m<=l do k:=k+1;l:=l-m;
```

precondition: $0 \leq n$

postconditions: $n = km + l \qquad 0 \leq l \qquad l < m$

proof goal: $\langle x_1 x_2 (r y_1 y_2)^* \neg r | p \leq q_1 q_2 \neg r$

Automating Hoare Logic

proof: two phases coupled by assignment rule $p[e/x] \leq |\{x := e\}]p$

1. MKA: goal follows from $p \leq |x_1]|x_2](q_1q_2)$ $\qquad q_1q_2r \leq |y_1]|y_2](q_1q_2)$
   (automated with Prover9)
2. arithmetics: subgoals must still be manually verified, e.g.,

$$\begin{aligned}|x_1]|x_2](q_1q_2) &= |\{k := 0\}]\ |\{l := n\}](q_1q_2) \geq (\{n = km + l\}\{0 \leq l\})[k/0][l/n] \\ &= \{n = 0m + n\}\{0 \leq n\} = \{0 \leq n\} \\ &= p\end{aligned}$$

remark:

- reasoning essentially inductive
- domain specific solvers should be integrated into ATPs
- try SPASS+T?

Newman's Lemma: A Proof Challenge

Newman's lemma: *A term rewriting system is confluent if it is locally confluent and terminating.*

generalisation and translation:

- x commutes over y $\quad y^*x^* \le x^*y^*$
- x locally commutes over y $\quad yx \le x^*y^*$

theorem: In ∇-Kleene algebra, if $x + y$ terminates and x locally commutes over y, then x commutes over y

Newman's Lemma: A Proof Challenge

proof: (so far)

- one page of semi-calculational arguments
- main calculation

$$\begin{aligned}\langle y^*|\langle y|\langle p\rangle|x\rangle|x^*\rangle &\le \langle y^*|\langle p_y\rangle\langle y||x\rangle\langle p_x\rangle|x^*\rangle\\ &\le \langle y^*|\langle p_y\rangle|x^*\rangle\langle y^*|\langle p_x\rangle|x^*\rangle\\ &\le \langle y^*|\langle p_y\rangle|x^*\rangle|x^*\rangle\langle y^*|\\ &\le \langle y^*|\langle p_y\rangle|x^*\rangle\langle y^*|\\ &\le |x^*\rangle\langle y^*|\langle y^*|\\ &\le |x^*\rangle\langle y^*|\end{aligned}$$

- $p_x = \langle x|p$ and $p_y = \langle y|p$
- proof lifted to level of modal operators

Newman's Lemma: A Proof Challenge

question: can you automate this?

remarks:

- Newman's lemma seems to require a mix of inductive and coinductive reasoning
- the main calculation mimics precisely the traditional diagrammatic proof
- more generally, Kleene algebras give an algebraic semantics to (some) rewrite diagrams

A Non-Modal Example: Back's Atomicity Refinement Law

demonic refinement algebra: [von Wright04] Kleene algebra

- with axiom $x0 = 0$ dropped
- extended by strong iteration $^\infty$ encompassing finite and infinite iteration

remark: abstracted from refinement calculus [BackvonWright]

atomicity refinement law for action systems

- complex theorem first published by Back in 1989
- long proof in set theory analysing infinite sequences
- proof by hand in demonic refinement algebra still covers 2 pages
- automated analysis reveals some glitches and yields generalisation

first task: build up library of verified basic refinement laws for proof

A Non-Modal Example: Back's Atomicity Refinement Law

theorem: if (i) $s \leq sq$ (ii) $a \leq qa$ (iii) $qb = 0$ (iv) $rb \leq br$
(v) $(a + r + b)l \leq l(a + r + b)$ (vi) $rq \leq qr$ (vii) $ql \leq lq$
(viii) $r^* = r^\infty$ (ix) $q \leq 1$

then

$$s(a + r + b + l)^\infty q \leq s(ab^\infty q + r + l)^\infty$$

two-step proof with "hypothesis learning"

1. assumptions imply $s(a + r + b + l)^\infty q \leq sl^\infty qr^\infty q(ab^\infty qr^\infty)^\infty$
   wait $60s$ for 75-step proof with Prover9
2. $q \leq 1$ implies $sl^\infty qr^\infty q(ab^\infty qr^\infty)^\infty \leq s(ab^\infty q + r + l)^\infty$
   wait $< 1s$ for 30-step proof

remark: full proof succeeds for $l = 0$ ($1013s$ for 46-step proof)

A Non-Modal Example: Back's Atomicity Refinement Law

equational proof can be reconstructed

$$\begin{aligned}
s(a+b+r+l)q &= sl^{\infty}(a+b+r)^{\infty}q \\
&= sl^{\infty}(b+r)^{\infty}(a(b+r)^{\infty})^{\infty}q \\
&= sl^{\infty}b^{\infty}r^{\infty}(ab^{\infty}r^{\infty})^{\infty}q \\
&\leq sl^{\infty}b^{\infty}r^{\infty}(qab^{\infty}r^{\infty})^{\infty}q \\
&= sl^{\infty}b^{\infty}r^{\infty}q(ab^{\infty}r^{\infty}q)^{\infty} \\
&\leq sql^{\infty}b^{\infty}r^{\infty}q(ab^{\infty}r^{\infty}q)^{\infty} \\
&\leq sl^{\infty}qb^{\infty}r^{\infty}q(ab^{\infty}r^{\infty}q)^{\infty} \\
&\leq sl^{\infty}qr^{\infty}q(ab^{\infty}r^{\infty}q)^{\infty} \\
&= sl^{\infty}qr^{\infty}q(ab^{\infty}r^{*}q)^{\infty} \\
&\leq sl^{\infty}qr^{\infty}q(ab^{\infty}qr^{*})^{\infty} \\
&= sl^{\infty}qr^{\infty}q(ab^{\infty}qr^{\infty})^{\infty}.
\end{aligned}$$

ATP Background

Ordered resolution: for ϕ maximal wrt syntactic ordering $\prec$ on terms/literals

$$\frac{\Gamma \to \Delta, \phi \quad \Gamma', \phi \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'} \qquad\qquad \frac{\Gamma \to \Delta, \phi, \phi}{\Gamma \to \Delta, \phi}$$

Redundancy: clause is $\prec$-redundant wrt clause set S if it is entailed by $\prec$-smaller instances of clauses from S

Orb: clause set closed under ordered resolution and redundancy elimination

Refutational completeness: orb of inconsistent clause set contains empty clause

remark: unification used at first-order level

ATP Background

strategy:

- transform first-oder formulas into clause set
- close working set under deduction rules
- apply deduction rules lazily
- apply redundancy elimination rules eagerly
- procedure must be fair with respect to clauses

ATP systems used:

- Prover9 and Vampire: fastest provers for algebraic theories
- Waldmeister: fastest tool for unit equations
- SPASS: ATP in sorted/typed setting

Conclusion

these lectures: modal Kleene algebras offer

- simple equational calculus including some (co)induction
- rich model class (traces, paths, languages, relations, functions,. . .)
- easy automation
- interesting applications in program analysis/verification
- relevant for modelling discrete dynamical systems

related work:

- automation of relation algebras similarly successful
- code at `www.dcs.shef.ac.uk/∼georg/ka`
- results will be integrated into TPTP library

Conclusion

general conclusion: ATP systems + computational algebras motivates

verification challenge

- off-the-shelf ATP with domain-specific algebras
- promising alternative to conventional approaches (model checking, HOL)
- light-weight formal methods with heavy-weight automation

Seek Simplicity and Distrust It.

[Whitehead]

Some References on Modal Kleene Algebras

1. *Prover9 and Mace4*, `http://www.cs.unm.edu/∼mccune/prover9`.
2. *Spass 3.0*, `http://spass.mpi-inf.mpg.de/`.
3. G. Struth. Modal Tools for Separation and Refinement Technical Report CS-08-07, Department of Computer Science; University of Sheffield, 2008.
4. J. Desharnais and G. Struth. Enabledness Conditions for Action Systems, Probabilistic Systems, and Processes Technical Report CS-08-06, Department of Computer Science; University of Sheffield, 2008.
5. J. Desharnais and G. Struth. Modal Semirings Revisited. (Accepted for MPC 2008).
6. B. Möller. Kleene getting lazy. *Science of Computer Programming*, 65(2):195–214, 2007.
7. L. Meinicke and K. Solin. Refinement algebra for probabilistic programs. In E. Boiten, J. Derrick, and G. Smith, editors, *Refine 2007*, ENTCS, 2007. To appear.
8. G. Struth. Reasoning Automatically about Termination and Refinement. In S. Ranise, editor, *6th International Workshop on First-Order Theorem Proving*, Technical Report ULCS-07-018, Department of Computer Science, University of Liverpool, pages 36–51. 2007.

9. P. Höfner and G. Struth. Automated Deduction in Kleene Algebra. In P. Pfenning, editor, *21th International Conference on Automated Deduction (CADE21)*, volume 4603 of *LNAI*, pages 279–294. Springer, 2007.
10. J.-L. De Carufel and J. Desharnais. Demonic algebra with domain. In R. A. Schmidt, editor, *Relations and Kleene Algebras in Computer Science*, volume 4136 of *LNCS*, pages 120–134. Springer, 2006.
11. K. Solin and J. von Wright. Refinement algebra with operators for enabledness and termination. In T. Uustalu, editor, *Mathematics of Program Construction*, volume 4014 of *LNCS*, pages 397–415. Springer, 2006.
12. P. Höfner, B. Möller and G. Struth. Quantales and temporal logics. In M. Johnson and V. Vene, editors, *Algebraic Methodology and Software Technology, 11th International Conference*, volume 4019 of *LNCS*, pages 262–277. Springer, 2006.
13. J. Desharnais, B. Möller, and G. Struth. Kleene algebra with domain. *ACM Transations on Computational Logic*, 7(4):798–833, 2006.
14. B. Möller, and G. Struth. Algebras of modal operators and partial correctness. *Theoretical Computer Science*, 351(2):221–239, 2006.
15. B. Möller and G. Struth. wp is wlp. In I. Düntsch, W. MacCaull and M. Winter, editors, *Relational Methods in Computer Science*, volume 3929 of *LNCS*, pages 200–211. Springer, 2005.

16. J. Desharnais, B. Möller, and G. Struth. Modal Kleene algebra and applications—a survey. *Journal on Relational Methods in Computer Science*, 1:93–131, 2004.(Invited contribution).

17. J. Desharnais, B. Möller, and G. Struth. Termination in modal Kleene algebra. In Jean-Jacques Lévy, Ernst W. Mayr, and John C. Mitchell, editors, *IFIP TCS2004*, pages 647–660. Kluwer, 2004. Revised version: *Algebraic Notions of Termination*. Technical Report 2006-23, Institut für Informatik, Universität Augsburg, 2006.