Monoidal Categories, Higher Categories

Jamie Vicary, University of Birmingham

Midlands Graduate School in the Foundations of Computing Science
University of Birmingham
14–18 April 2019
Overview

Welcome to the course! We will cover the following topics:
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
IV. Duality for monoidal categories
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
IV. Duality for monoidal categories
V. Duality for monoidal 2-categories
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
IV. Duality for monoidal categories
V. Duality for monoidal 2-categories

In the third lecture, we will use the proof assistant http://homotopy.io to work directly with higher categorical objects. Bring a laptop!
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
IV. Duality for monoidal categories
V. Duality for monoidal 2-categories

In the third lecture, we will use the proof assistant http://homotopy.io to work directly with higher categorical objects. Bring a laptop!

There are no separate exercise classes, but there will be some interesting problems to look at together along the way.
Overview

Welcome to the course! We will cover the following topics:

I. Monoidal categories
II. Higher categories
III. Coherence for monoidal categories
IV. Duality for monoidal categories
V. Duality for monoidal 2-categories

In the third lecture, we will use the proof assistant http://homotopy.io to work directly with higher categorical objects. Bring a laptop!

There are no separate exercise classes, but there will be some interesting problems to look at together along the way.

Examples will be drawn from sets, relations, and Hilbert spaces, giving insight into applications to classical, nondeterministic, and quantum computation.
Part I

Monoidal categories
I. Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.
I. Monoidal categories

Category theory describes systems and processes:

- physical systems, and physical processes governing them;
- data types, and algorithms manipulating them;
- algebraic structures, and structure-preserving functions;
- logical propositions, and implications between them.

Monoidal category theory adds the idea of parallelism:

- independent physical systems evolve simultaneously;
- running computer algorithms in parallel;
- products or sums of algebraic or geometric structures;
- using separate proofs of $P$ and $Q$ to construct a proof of the conjunction ($P$ and $Q$).
I. Monoidal categories

Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition
I. Monoidal categories

Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these processes?

\[(A \otimes B) \otimes C \quad A \otimes (B \otimes C)\]
I. Monoidal categories

Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these processes?

$$\begin{align*}
  (A \otimes B) \otimes C &= A \otimes (B \otimes C) \\
  (S \times T) \times U &\neq S \times (T \times U).
\end{align*}$$

- It’s not right to say they’re *equal*, since even just for sets,
Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition
- What *relationship* should there be between these processes?

\[(A \otimes B) \otimes C \quad A \otimes (B \otimes C)\]

- It’s not right to say they’re *equal*, since even just for sets,

\[(S \times T) \times U \neq S \times (T \times U)\]

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
I. Monoidal categories

Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition.
- What *relationship* should there be between these processes?
  
  $$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

- It’s not right to say they’re *equal*, since even just for sets,
  
  $$(S \times T) \times U \neq S \times (T \times U).$$

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
I. Monoidal categories

Why should this theory be interesting?

- Let $A$, $B$ and $C$ be processes, and let $\otimes$ be parallel composition.
- What *relationship* should there be between these processes?

\[(A \otimes B) \otimes C \neq A \otimes (B \otimes C)\]

- It’s not right to say they’re *equal*, since even just for sets,

\[(S \times T) \times U \neq S \times (T \times U).\]

- Maybe they should be *isomorphic* — but then what *equations* should these isomorphisms satisfy?
- How do we treat *trivial* systems?
- What should the relationship be between $A \otimes B$ and $B \otimes A$?
I. Monoidal categories

Definition 1. A monoidal category is a category $\mathbf{C}$ equipped with the following data:
I. Monoidal categories

Definition 1. A monoidal category is a category $C$ equipped with the following data:

- a tensor product functor

$$\otimes : C \times C \rightarrow C;$$
I. Monoidal categories

Definition 1. A monoidal category is a category $\mathcal{C}$ equipped with the following data:

- a tensor product functor

$$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C};$$

- a unit object

$$I \in \text{Ob}(\mathcal{C});$$
I. Monoidal categories

Definition 1. A monoidal category is a category $\mathbf{C}$ equipped with the following data:

- a tensor product functor

\[ \otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}; \]

- a unit object

\[ I \in \text{Ob}(\mathbf{C}); \]

- a family of associator natural isomorphisms

\[ (A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C); \]
I. Monoidal categories

Definition 1. A monoidal category is a category $\mathcal{C}$ equipped with the following data:

- a tensor product functor
  \[ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}; \]
- a unit object
  \[ I \in \text{Ob}(\mathcal{C}); \]
- a family of associator natural isomorphisms
  \[ (A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C); \]
- a family of left unitor natural isomorphisms
  \[ I \otimes A \xrightarrow{\lambda_A} A; \]
I. Monoidal categories

Definition 1. A monoidal category is a category \( \mathcal{C} \) equipped with the following data:

- a tensor product functor

\[ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}; \]

- a unit object

\[ I \in \text{Ob}(\mathcal{C}); \]

- a family of associator natural isomorphisms

\[ (A \otimes B) \otimes C \cong A \otimes (B \otimes C); \]

- a family of left unitor natural isomorphisms

\[ I \otimes A \cong A; \]

- and a family of right unitor natural isomorphisms

\[ A \otimes I \cong A. \]
I. Monoidal categories

This data must satisfy the *triangle* and *pentagon* equations, for all objects $A$, $B$, $C$ and $D$:

\[(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B)\]

\[
\rho_A \otimes \text{id}_B \quad A \otimes B \quad \text{id}_A \otimes \lambda_B
\]
I. Monoidal categories

This data must satisfy the triangle and pentagon equations, for all objects $A$, $B$, $C$ and $D$:

$$
\begin{array}{c}
(A \otimes I) \otimes B \\
\downarrow \alpha_{A,I,B} \\
A \otimes (I \otimes B)
\end{array}
\begin{array}{c}
\rho_A \otimes \text{id}_B \\
\uparrow \\
A \otimes B
\end{array}
\begin{array}{c}
\text{id}_A \otimes \lambda_B \\
\downarrow \\
\end{array}

\begin{array}{c}
(A \otimes (B \otimes C)) \otimes D \\
\downarrow \alpha_{A,B \otimes C,D} \\
A \otimes ((B \otimes C) \otimes D)
\end{array}
\begin{array}{c}
\alpha_{A,B,C} \otimes \text{id}_D \\
\uparrow \\
((A \otimes B) \otimes C) \otimes D
\end{array}
\begin{array}{c}
\text{id}_A \otimes \alpha_{B,C,D} \\
\downarrow \\
A \otimes (B \otimes (C \otimes D))
\end{array}
\begin{array}{c}
\alpha_{A \otimes B,C,D} \\
\rightarrow \\
(A \otimes B) \otimes (C \otimes D)
\end{array}
\begin{array}{c}
\alpha_{A,B,C \otimes D} \\
\rightarrow
\end{array}
\begin{array}{c}
(A \otimes B) \otimes (C \otimes D)
\end{array}
I. Monoidal categories

This data must satisfy the triangle and pentagon equations, for all objects $A$, $B$, $C$ and $D$:

$$
(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B)
$$

$$
\rho_A \otimes \text{id}_B \quad \text{id}_A \otimes \lambda_B
\quad A \otimes B
$$

$$
(A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D)
$$

$$
\alpha_{A,B,C} \otimes \text{id}_D
\uparrow
(A \otimes B) \otimes C \otimes D
\downarrow \text{id}_A \otimes \alpha_{B,C,D}
\quad A \otimes (B \otimes (C \otimes D))
$$

$$
\alpha_{A \otimes B,C,D}
\quad (A \otimes B) \otimes (C \otimes D)
\quad \alpha_{A,B,C \otimes D}
$$

**Theorem 2.** If the pentagon and triangle equations hold, then so does any well-typed equation built from $\alpha$, $\lambda$, $\rho$ and their inverses.
I. Monoidal categories

This data must satisfy the triangle and pentagon equations, for all objects $A$, $B$, $C$ and $D$:

\[
\begin{align*}
(A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B) \\
\rho_A \otimes \text{id}_B & \quad \text{id}_A \otimes \lambda_B
\end{align*}
\]

\[
\begin{align*}
(A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B \otimes C,D}} A \otimes ((B \otimes C) \otimes D) \\
\alpha_{A,B,C} \otimes \text{id}_D & \quad \text{id}_A \otimes \alpha_{B,C,D}
\end{align*}
\]

\[
\begin{align*}
(A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} A \otimes (B \otimes (C \otimes D)) \\
\alpha_{A \otimes B,C,D} & \quad \alpha_{A,B,C \otimes D}
\end{align*}
\]

**Theorem 2.** If the pentagon and triangle equations hold, then so does any well-typed equation built from $\alpha$, $\lambda$, $\rho$ and their inverses.

**Exercise.** Use the triangle and pentagon equations to prove $\lambda_I = \rho_I$. 
I. Monoidal categories

The monoidal structure on $\textbf{Set}$ is given by Cartesian product.
I. Monoidal categories

The monoidal structure on $\text{Set}$ is given by Cartesian product.

**Definition 3.** The monoidal structure on the category $\text{Set}$, and also by restriction on $\text{FSet}$, is defined as follows:
I. Monoidal categories

The monoidal structure on Set is given by Cartesian product.

Definition 3. The monoidal structure on the category Set, and also by restriction on FSet, is defined as follows:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = (f(a); g(c))$
I. Monoidal categories

The monoidal structure on \( \textbf{Set} \) is given by Cartesian product.

**Definition 3.** The monoidal structure on the category \( \textbf{Set} \), and also by restriction on \( \textbf{FSet} \), is defined as follows:

- **the tensor product** is Cartesian product of sets, written \( \times \), acting on functions \( A \xrightarrow{f} B \) and \( C \xrightarrow{g} D \) as \((f \times g)(a, c) = (f(a); g(c))\).
- **the unit object** is a chosen singleton set \( \{ \bullet \} \).
The monoidal structure on $\textbf{Set}$ is given by Cartesian product.

**Definition 3.** The monoidal structure on the category $\textbf{Set}$, and also by restriction on $\textbf{FSet}$, is defined as follows:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = (f(a); g(c))$

- **the unit object** is a chosen singleton set $\{\bullet\}$;

- **associators** $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the functions given by $((a, b), c) \mapsto (a, (b, c))$;
The monoidal structure on $\textbf{Set}$ is given by Cartesian product.

**Definition 3.** The monoidal structure on the category $\textbf{Set}$, and also by restriction on $\textbf{FSet}$, is defined as follows:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = (f(a); g(c))$.
- **the unit object** is a chosen singleton set $\{\bullet\}$;
- **associators** $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the functions given by $((a, b), c) \mapsto (a, (b, c))$;
- **left unitors** $I \times A \xrightarrow{\lambda_A} A$ are the functions $(\bullet, a) \mapsto a$;
I. Monoidal categories

The monoidal structure on \textbf{Set} is given by Cartesian product.

\textbf{Definition 3.} The monoidal structure on the category \textbf{Set}, and also by restriction on \textbf{FSet}, is defined as follows:

- \textbf{the tensor product} is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = (f(a); g(c))$

- \textbf{the unit object} is a chosen singleton set $\{\bullet\}$;

- \textbf{associators} $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the functions given by $((a, b), c) \mapsto (a, (b, c))$;

- \textbf{left unitors} $I \times A \xrightarrow{\lambda_A} A$ are the functions $(\bullet, a) \mapsto a$;

- \textbf{right unitors} $A \times I \xrightarrow{\rho_A} A$ are the functions $(a, \bullet) \mapsto a$. 
The monoidal structure on $\textbf{Set}$ is given by Cartesian product.

**Definition 3.** The monoidal structure on the category $\textbf{Set}$, and also by restriction on $\textbf{FSet}$, is defined as follows:

- **the tensor product** is Cartesian product of sets, written $\times$, acting on functions $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ as $(f \times g)(a, c) = (f(a); g(c))$
- **the unit object** is a chosen singleton set $\{\bullet\}$;
- **associators** $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the functions given by $((a, b), c) \mapsto (a, (b, c))$;
- **left unitors** $I \times A \xrightarrow{\lambda_A} A$ are the functions $(\bullet, a) \mapsto a$;
- **right unitors** $A \times I \xrightarrow{\rho_A} A$ are the functions $(a, \bullet) \mapsto a$.

Other tensor products exist, but this one plays a canonical role in our interpretation of classical reality.
I. Monoidal categories

**Definition 4.** The category $\textbf{Hilb}$ has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory $\textbf{FHilb}$ is its restriction to finite-dimensional Hilbert spaces.
I. Monoidal categories

Definition 4. The category $\text{Hilb}$ has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory $\text{FHilb}$ is its restriction to finite-dimensional Hilbert spaces.

Definition 5. The monoidal structure on the category $\text{Hilb}$, and also by restriction on $\text{FHilb}$, is defined in the following way:
I. Monoidal categories

**Definition 4.** The category $\text{Hilb}$ has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory $\text{FHilb}$ is its restriction to finite-dimensional Hilbert spaces.

**Definition 5.** The monoidal structure on the category $\text{Hilb}$, and also by restriction on $\text{FHilb}$, is defined in the following way:

- the tensor product $\otimes : \text{Hilb} \times \text{Hilb} \to \text{Hilb}$ is the *tensor product* of Hilbert spaces;
I. Monoidal categories

**Definition 4.** The category $\text{Hilb}$ has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory $\text{FHilb}$ is its restriction to finite-dimensional Hilbert spaces.

**Definition 5.** The monoidal structure on the category $\text{Hilb}$, and also by restriction on $\text{FHilb}$, is defined in the following way:

- **the tensor product** $\otimes: \text{Hilb} \times \text{Hilb} \to \text{Hilb}$ is the *tensor product* of Hilbert spaces;
- **the unit object** $I$ is the one-dimensional Hilbert space $\mathbb{C}$;
I. Monoidal categories

Definition 4. The category \textbf{Hilb} has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory \textbf{FHilb} is its restriction to finite-dimensional Hilbert spaces.

Definition 5. The monoidal structure on the category \textbf{Hilb}, and also by restriction on \textbf{FHilb}, is defined in the following way:

- the tensor product \( \otimes : \text{Hilb} \times \text{Hilb} \to \text{Hilb} \) is the tensor product of Hilbert spaces;
- the unit object \( I \) is the one-dimensional Hilbert space \( \mathbb{C} \);
- associators \( (H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K) \) are the unique linear maps satisfying \( (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \) for all \( u \in H, v \in J \) and \( w \in K \);
I. Monoidal categories

Definition 4. The category \( \text{Hilb} \) has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory \( \text{FHilb} \) is its restriction to finite-dimensional Hilbert spaces.

Definition 5. The monoidal structure on the category \( \text{Hilb} \), and also by restriction on \( \text{FHilb} \), is defined in the following way:

- the tensor product \( \otimes : \text{Hilb} \times \text{Hilb} \rightarrow \text{Hilb} \) is the tensor product of Hilbert spaces;
- the unit object \( I \) is the one-dimensional Hilbert space \( \mathbb{C} \);
- associators \( (H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K) \) are the unique linear maps satisfying \( (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \) for all \( u \in H, v \in J \) and \( w \in K \);
- left unitors \( \mathbb{C} \otimes H \xrightarrow{\lambda_H} H \) are the unique linear maps satisfying \( 1 \otimes u \mapsto u \) for all \( u \in H \);
I. Monoidal categories

Definition 4. The category $\text{Hilb}$ has objects given by Hilbert spaces, and morphisms given by bounded linear maps. The subcategory $\text{FHilb}$ is its restriction to finite-dimensional Hilbert spaces.

Definition 5. The monoidal structure on the category $\text{Hilb}$, and also by restriction on $\text{FHilb}$, is defined in the following way:

- **the tensor product** $\otimes : \text{Hilb} \times \text{Hilb} \rightarrow \text{Hilb}$ is the *tensor product* of Hilbert spaces;
- **the unit object** $I$ is the one-dimensional Hilbert space $\mathbb{C}$;
- **associators** $(H \otimes J) \otimes K \xrightarrow{\alpha_{H,J,K}} H \otimes (J \otimes K)$ are the unique linear maps satisfying $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ for all $u \in H$, $v \in J$ and $w \in K$;
- **left unitors** $\mathbb{C} \otimes H \xrightarrow{\lambda_H} H$ are the unique linear maps satisfying $1 \otimes u \mapsto u$ for all $u \in H$;
- **right unitors** $H \otimes \mathbb{C} \xrightarrow{\rho_H} H$ are the unique linear maps satisfying $u \otimes 1 \mapsto u$ for all $u \in H$. 
I. Monoidal categories

Relations give another notion of process between sets.
I. Monoidal categories

Relations give another notion of process between sets.

**Definition 6.** Given sets $A$ and $B$, a relation $A \xrightarrow{R} B$ is a subset $R \subseteq A \times B$. 
I. Monoidal categories

Relations give another notion of process between sets.

**Definition 6.** Given sets $A$ and $B$, a relation $A \xrightarrow{R} B$ is a subset $R \subseteq A \times B$.

We can think of a relation $A \xrightarrow{R} B$ in a dynamical way, as specifying how states of $A$ can evolve into states of $B$:

\[ A \xrightarrow{R} B \]
I. Monoidal categories

Relations give another notion of process between sets.

Definition 6. Given sets $A$ and $B$, a relation $A \xrightarrow{R} B$ is a subset $R \subseteq A \times B$.

We can think of a relation $A \xrightarrow{R} B$ in a dynamical way, as specifying how states of $A$ can evolve into states of $B$:

This is nondeterministic, because an element of $A$ can be related to more than one element of $B$, or to none.
I. Monoidal categories

Suppose we have a pair of head-to-tail relations:

\[ A \xrightarrow{R} B \quad \text{and} \quad B \xrightarrow{S} C \]
I. Monoidal categories

Suppose we have a pair of head-to-tail relations:

\[ A \xrightarrow{R} B \quad B \xrightarrow{S} C \]

Then our interpretation gives a natural notion of composition:

\[ A \xrightarrow{S \circ R} C \]
We can write relations as (0,1)-valued matrices:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Composition of relations is then ordinary matrix multiplication, with logical disjunction (OR) and conjunction (AND) for + and \( \times \).
I. Monoidal categories

The intuition we have developed leads to the following definition of the category \textbf{Rel}.
I. Monoidal categories

The intuition we have developed leads to the following definition of the category $\text{Rel}$.

**Definition 7.** The category $\text{Rel}$ of sets and relations is defined as follows:
I. Monoidal categories

The intuition we have developed leads to the following definition of the category \textbf{Rel}.

\textbf{Definition 7.} The category \textbf{Rel} of sets and relations is defined as follows:

- \textbf{objects} are sets $A, B, C, \ldots$;
I. Monoidal categories

The intuition we have developed leads to the following definition of the category \textbf{Rel}.

**Definition 7.** The category \textbf{Rel} of sets and relations is defined as follows:

- **objects** are sets $A, B, C, \ldots$;
- **morphisms** are relations $R \subseteq A \times B$, with $(a, b) \in R$ written $aRb$;
I. Monoidal categories

The intuition we have developed leads to the following definition of the category \( \text{Rel} \).

**Definition 7.** The category \( \text{Rel} \) of sets and relations is defined as follows:

- **objects** are sets \( A, B, C, \ldots \);
- **morphisms** are relations \( R \subseteq A \times B \), with \((a, b) \in R\) written \( aRb \);
- **composition** of \( A \xrightarrow{R} B \) and \( B \xrightarrow{S} C \) is the relation
  \[ \{(a, c) \in A \times C \mid \exists b \in B : aRb, bSc\} \].
I. Monoidal categories

The intuition we have developed leads to the following definition of the category $\text{Rel}$.

**Definition 7.** The category $\text{Rel}$ of sets and relations is defined as follows:

- **objects** are sets $A, B, C, \ldots$;
- **morphism**s are relations $R \subseteq A \times B$, with $(a, b) \in R$ written $aRb$;
- **composition** of $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ is the relation
  $$\{(a, c) \in A \times C \mid \exists b \in B : aRb, bSc\};$$
- **the identity morphism** on $A$ is the relation
  $$\{(a, a) \in A \times A \mid a \in A\}.$$

Define the category $\text{FRel}$ to be the restriction of $\text{Rel}$ to finite sets.
I. Monoidal categories

The intuition we have developed leads to the following definition of the category $\text{Rel}$.

**Definition 7.** The category $\text{Rel}$ of sets and relations is defined as follows:

- **objects** are sets $A, B, C, \ldots$;
- **morphism**s are relations $R \subseteq A \times B$, with $(a, b) \in R$ written $aRb$;
- **composition** of $A \xrightarrow{R} B$ and $B \xrightarrow{S} C$ is the relation $\{(a, c) \in A \times C \mid \exists b \in B : aRb, bSc\}$;
- **the identity morphism** on $A$ is the relation $\{(a, a) \in A \times A \mid a \in A\}$.

Define the category $\text{FRel}$ to be the restriction of $\text{Rel}$ to finite sets.

While $\text{Set}$ is a setting for classical physics, and $\text{Hilb}$ is a setting for quantum physics, $\text{Rel}$ is somewhere in the middle.

It seems like $\text{Rel}$ should be a lot like $\text{Set}$, but we will discover it behaves a lot more like $\text{Hilb}$. 
I. Monoidal categories

There is a canonical monoidal structure on the category $\text{Rel}$. 
I. Monoidal categories

There is a canonical monoidal structure on the category \textbf{Rel}.

**Definition 8.** The monoidal structure on the category \textbf{Rel} is defined in the following way:

- **the tensor product** is Cartesian product of sets, written \( \times \), acting on relations \( A \xrightarrow{R} B \) and \( C \xrightarrow{S} D \) by setting \((a, c)(R \times S)(b, d)\) if and only if \(aRb\) and \(cSd\);
I. Monoidal categories

There is a canonical monoidal structure on the category \textbf{Rel}.

\textbf{Definition 8.} The monoidal structure on the category \textbf{Rel} is defined in the following way:

- \textbf{the tensor product} is Cartesian product of sets, written \( \times \), acting on relations \( A \xrightarrow{R} B \) and \( C \xrightarrow{S} D \) by setting \( (a, c)(R \times S)(b, d) \) if and only if \( aRb \) and \( cSd \);

- \textbf{the unit object} is a chosen singleton set \( = \{ \bullet \} \);
I. Monoidal categories

There is a canonical monoidal structure on the category \( \textbf{Rel} \).

**Definition 8.** The monoidal structure on the category \( \textbf{Rel} \) is defined in the following way:

- **the tensor product** is Cartesian product of sets, written \( \times \), acting on relations \( A \xrightarrow{R} B \) and \( C \xrightarrow{S} D \) by setting \((a, c)(R \times S)(b, d)\) if and only if \( aRb \) and \( cSd \);
- **the unit object** is a chosen singleton set \( \{\bullet\} \);
- **associators** \( (A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C) \) are the relations defined by \( ((a, b), c) \sim (a, (b, c)) \);
I. Monoidal categories

There is a canonical monoidal structure on the category \( \text{Rel} \).

**Definition 8.** The monoidal structure on the category \( \text{Rel} \) is defined in the following way:

- **the tensor product** is Cartesian product of sets, written \( \times \), acting on relations \( A \xrightarrow{R} B \) and \( C \xrightarrow{S} D \) by setting \((a, c)(R \times S)(b, d)\) if and only if \(aRb\) and \(cSd\);

- **the unit object** is a chosen singleton set \( = \{\bullet\} \);

- **associators** \((A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)\) are the relations defined by \(((a, b), c) \sim (a, (b, c))\);

- **left unitors** \(I \times A \xrightarrow{\lambda_A} A\) are the relations defined by \((\bullet, a) \sim a\);
I. Monoidal categories

There is a canonical monoidal structure on the category \( \text{Rel} \).

**Definition 8.** The monoidal structure on the category \( \text{Rel} \) is defined in the following way:

- **the tensor product** is Cartesian product of sets, written \( \times \), acting on relations \( A \xrightarrow{R} B \) and \( C \xrightarrow{S} D \) by setting \((a, c)(R \times S)(b, d)\) if and only if \(aRb\) and \(cSd\);
- **the unit object** is a chosen singleton set \( \{\bullet\} \);
- **associators** \((A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)\) are the relations defined by \(((a, b), c) \sim (a, (b, c))\);
- **left unitors** \( I \times A \xrightarrow{\lambda_A} A \) are the relations defined by \((\bullet, a) \sim a\);
- **right unitors** \( A \times I \xrightarrow{\rho_A} A \) are the relations defined by \((a, \bullet) \sim a\).
I. Monoidal categories

There is a canonical monoidal structure on the category \textbf{Rel}.

\textbf{Definition 8.} The monoidal structure on the category \textbf{Rel} is defined in the following way:

- \textbf{the tensor product} is Cartesian product of sets, written $\times$, acting on relations $A \xrightarrow{R} B$ and $C \xrightarrow{S} D$ by setting $(a, c)(R \times S)(b, d)$ if and only if $aRb$ and $cSd$;
- \textbf{the unit object} is a chosen singleton set $= \{\bullet\}$;
- \textbf{associators} $(A \times B) \times C \xrightarrow{\alpha_{A,B,C}} A \times (B \times C)$ are the relations defined by $((a, b), c) \sim (a, (b, c))$;
- \textbf{left unitors} $I \times A \xrightarrow{\lambda_A} A$ are the relations defined by $(\bullet, a) \sim a$;
- \textbf{right unitors} $A \times I \xrightarrow{\rho_A} A$ are the relations defined by $(a, \bullet) \sim a$.

The Cartesian product is \textit{not} a categorical product in \textbf{Rel}, so although this monoidal structure looks like that of \textbf{Set}, it is more similar to the structure on \textbf{Hilb}. 
I. Monoidal categories

Monoidal categories satisfy the *interchange law*, which governs the interaction between composition and tensor product.
I. Monoidal categories

Monoidal categories satisfy the *interchange law*, which governs the interaction between composition and tensor product.

**Theorem 9.** Any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ and $E \xrightarrow{j} F$ in a monoidal category satisfy the interchange law:

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$
I. Monoidal categories

Monoidal categories satisfy the *interchange law*, which governs the interaction between composition and tensor product.

**Theorem 9.** Any morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{h} E$ and $E \xrightarrow{j} F$ in a monoidal category satisfy the interchange law:

$$(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)$$

**Proof.** This holds because of properties of the category $\mathbf{C} \times \mathbf{C}$, and from the fact that $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a functor:

$$(g \circ f) \otimes (j \circ h) \equiv \otimes(g \circ f, j \circ h)$$
$$= \otimes((g,j) \circ (f,h)) \quad \text{(composition in } \mathbf{C} \times \mathbf{C})$$
$$= (\otimes(g,j)) \circ (\otimes(f,h)) \quad \text{(functoriality of } \otimes)$$
$$= (g \otimes j) \circ (f \otimes h)$$

Remember the functoriality property: $F(g \circ f) = F(g) \circ F(f)$. 
I. Monoidal categories

Monoidal categories have an elegant graphical calculus.
I. Monoidal categories

Monoidal categories have an elegant graphical calculus.

For morphisms \( A \xrightarrow{f} B \) and \( C \xrightarrow{g} D \), we draw their tensor product \( A \otimes C \xrightarrow{f \otimes g} B \otimes D \) like this:

\[
\begin{array}{c}
\begin{array}{c}
B \\
\hline
f \\
A
\end{array} & & &
\begin{array}{c}
D \\
\hline
g \\
C
\end{array}
\end{array}
\]

The idea is that \( f \) and \( g \) represent distinct processes taking place at the same time.
Monoidal categories have an elegant graphical calculus.

For morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, we draw their tensor product $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ like this:

```
     B   D
    |    |
  f   |   g
    |    |
A   C
```

The idea is that $f$ and $g$ represent distinct processes taking place at the same time.

Inputs are drawn at the bottom, and outputs are drawn at the top; in this sense, “time” runs upwards.
I. Monoidal categories

The monoidal unit object $I$ is drawn as the empty diagram:
I. Monoidal categories

The monoidal unit object $I$ is drawn as the empty diagram:

The left unitor $I \otimes A \xrightarrow{\lambda_A} A$, the right unitor $A \otimes I \xrightarrow{\rho_A} A$ and the associator $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$ are also not depicted:
I. Monoidal categories

The monoidal unit object $I$ is drawn as the empty diagram:

The left unitor $I \otimes A \xrightarrow{\lambda_A} A$, the right unitor $A \otimes I \xrightarrow{\rho_A} A$ and the associator $(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$ are also not depicted:

\[
\begin{array}{ccc}
A & A & A \\
\lambda_A & \rho_A & \alpha_{A,B,C}
\end{array}
\]

The coherence of $\alpha$, $\lambda$ and $\rho$ is essential for the graphical calculus to function. Since there can only be a single morphism built from their components of any given type, it doesn’t matter that their graphical calculus encodes no information.
I. Monoidal categories

Now let’s look at the interchange law:

\[(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)\]
I. Monoidal categories

Now let’s look at the interchange law:

\[(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)\]

Graphically it’s trivial.
I. Monoidal categories

Now let’s look at the interchange law:

\[(g \circ f) \otimes (j \circ h) = (g \otimes j) \circ (f \otimes h)\]

Graphically it’s trivial.

The apparent complexity of the theory of monoidal categories—\(\alpha, \lambda, \rho,\) coherence, interchange—was in fact complexity of the geometry of the plane. So when we use a geometrical notation, the complexity vanishes.
I. Monoidal categories

Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
- components of the diagram never intersect.
I. Monoidal categories

Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
- components of the diagram never intersect.

Here are examples of isotopic and non-isotopic diagrams:
I. Monoidal categories

Two diagrams are *planar isotopic* when one can be deformed continuously into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
- components of the diagram never intersect.

Here are examples of isotopic and non-isotopic diagrams:

We will allow heights of the diagrams to change, and allow input and output wires to slide horizontally along the boundary, although they must never change order.
I. Monoidal categories

We can now state the correctness theorem.

**Theorem 10.** A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.
I. Monoidal categories

We can now state the correctness theorem.

**Theorem 10.** A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

Let $f$ and $g$ be morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

- $P(f, g) = \text{‘under the axioms of a monoidal category, } f = g\text{’}$
- $Q(f, g) = \text{‘graphically, } f \text{ and } g \text{ are planar isotopic\’}$
I. Monoidal categories

We can now state the correctness theorem.

**Theorem 10.** A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

Let $f$ and $g$ be morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

- $P(f, g) = \text{‘under the axioms of a monoidal category, } f = g\text{’}$
- $Q(f, g) = \text{‘graphically, } f \text{ and } g \text{ are planar isotopic’}$

**Soundness** is the assertion that for all such $f$ and $g$, $P(f, g) \Rightarrow Q(f, g)$. It is easy to prove: just check each axiom.
I. Monoidal categories

We can now state the correctness theorem.

**Theorem 10.** A well-formed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

Let \( f \) and \( g \) be morphisms such that the equation \( f = g \) is well-formed, and consider the following statements:

- \( P(f, g) = \) ‘under the axioms of a monoidal category, \( f = g \)’
- \( Q(f, g) = \) ‘graphically, \( f \) and \( g \) are planar isotopic’

*Soundness* is the assertion that for all such \( f \) and \( g \), \( P(f, g) \Rightarrow Q(f, g) \). It is easy to prove: just check each axiom.

*Completeness* is the reverse assertion, that for all such \( f \) and \( g \), \( Q(f, g) \Rightarrow P(f, g) \). It is hard to prove; one must show that planar isotopy is generated by a finite set of moves, each being implied by the monoidal axioms.
I. Monoidal categories

In a category, we cannot ‘look inside’ an object to inspect its elements. We have do everything using the morphisms.
I. Monoidal categories

In a category, we cannot ‘look inside’ an object to inspect its elements. We have to do everything using the morphisms.

**Definition 11.** In a monoidal category, a *state* of an object $A$ is a morphism $I \to A$. 
I. Monoidal categories

In a category, we cannot ‘look inside’ an object to inspect its elements. We have to do everything using the morphisms.

**Definition 11.** In a monoidal category, a *state* of an object $A$ is a morphism $I \to A$.

The monoidal unit object represents the trivial system, so a state is a way for the system $A$ to be ‘brought into existence’.
I. Monoidal categories

In a category, we cannot ‘look inside’ an object to inspect its elements. We have do everything using the morphisms.

**Definition 11.** In a monoidal category, a *state* of an object $A$ is a morphism $I \rightarrow A$.

The monoidal unit object represents the trivial system, so a state is a way for the system $A$ to be ‘brought into existence’.

We draw a state $I \xrightarrow{a} A$ like this:

\[
\begin{array}{c}
A \\
\downarrow \quad a \\
\end{array}
\]
Example 12. Let’s examine the states in our example categories.

- In **Hilb**, states of a Hilbert space $H$ are linear functions $\mathbb{C} \to H$, which correspond to *elements* of $H$ by considering the image of $1 \in \mathbb{C}$.
Example 12. Let’s examine the states in our example categories.

- In \textbf{Hilb}, states of a Hilbert space \( H \) are linear functions \( \mathbb{C} \to H \), which correspond to \textit{elements} of \( H \) by considering the image of \( 1 \in \mathbb{C} \).

- In \textbf{Set}, states of a set \( A \) are functions \( \{\bullet\} \to A \), which correspond to \textit{elements} of \( A \) by considering the image of \( \bullet \).
Example 12. Let’s examine the states in our example categories.

- In \textbf{Hilb}, states of a Hilbert space \( H \) are linear functions \( \mathbb{C} \to H \), which correspond to \textit{elements} of \( H \) by considering the image of \( 1 \in \mathbb{C} \).

- In \textbf{Set}, states of a set \( A \) are functions \( \{\bullet\} \to A \), which correspond to \textit{elements} of \( A \) by considering the image of \( \bullet \).

- In \textbf{Rel}, states of a set \( A \) are relations \( \{\bullet\} \xrightarrow{R} A \), which correspond to \textit{subsets} by considering all elements related to \( \bullet \).
The dual notion of state is effect.

**Definition 13.** In a monoidal category, an *effect* on an object $A$ is a morphism $A \rightarrow I$. 
The dual notion of state is effect.

**Definition 13.** In a monoidal category, an *effect* on an object $A$ is a morphism $A \to I$.

We can use states, effects and other morphisms to build up interesting diagrams, which give ‘histories’ for a family of systems:

We can interpret an effect as a *property observation* of a system. Overall this composite gives a state of $A$. 
I. Monoidal categories

A morphism $I \rightarrow c \rightarrow A \otimes B$ is a *joint state* of $A$ and $B$. We depict it graphically in the following way.

```
   A  B
  /|
 / |
/  |
|   |
c
```
I. Monoidal categories

A morphism $I \xrightarrow{c} A \otimes B$ is a joint state of $A$ and $B$. We depict it graphically in the following way.

\[
\begin{array}{c}
A \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad c \\
\end{array} 
\quad B
\]

Definition 14. A joint state $I \xrightarrow{c} A \otimes B$ is a product state when it is of the form $I \xrightarrow{\lambda_I} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:

\[
\begin{array}{ccc}
\begin{array}{c}
A \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad c \\
\end{array} 
\quad B & = & A \\
\downarrow \\
\downarrow \\
\downarrow \\
\quad a \\
\end{array} 
\quad B
\]
\]
I. Monoidal categories

A morphism $I \rightarrow A \otimes B$ is a joint state of $A$ and $B$. We depict it graphically in the following way.

Definition 14. A joint state $I \rightarrow A \otimes B$ is a product state when it is of the form $I \xrightarrow{\lambda_I} I \otimes I \xrightarrow{a \otimes b} A \otimes B$:

Definition 15. A joint state is entangled when it is not a product state.
I. Monoidal categories

Example 16. Let’s investigate joint states, product states, and entangled states in our example categories.

- In Hilb:
  - joint states of $H$ and $K$ are elements of $H \otimes K$;
  - product states are factorizable states;
  - entangled states are elements of $H \otimes K$ which cannot be factorized, i.e. entangled states in the quantum sense.
I. Monoidal categories

Example 16. Let’s investigate joint states, product states, and entangled states in our example categories.

- In Hilb:
  - **joint states** of $H$ and $K$ are elements of $H \otimes K$;
  - **product states** are factorizable states;
  - **entangled states** are elements of $H \otimes K$ which cannot be factorized, i.e. entangled states in the quantum sense.

- In Set:
  - **joint states** of $A$ and $B$ are elements of $A \times B$;
  - **product states** are elements $(a, b) \in A \times B$;
  - **entangled states** don’t exist.
Example 16. Let’s investigate joint states, product states, and entangled states in our example categories.

- In Hilb:
  - **joint states** of $H$ and $K$ are elements of $H \otimes K$;
  - **product states** are factorizable states;
  - **entangled states** are elements of $H \otimes K$ which cannot be factorized, i.e. entangled states in the quantum sense.

- In Set:
  - **joint states** of $A$ and $B$ are elements of $A \times B$;
  - **product states** are elements $(a, b) \in A \times B$;
  - **entangled states** don’t exist.

- In Rel:
  - **joint states** of $A$ and $B$ are subsets of $A \times B$;
  - **product states** are subsets $U \subseteq A \times B$ such that, for some $V \subseteq A$ and $W \subseteq B$, $(v, w) \in U$ if and only if $v \in V$, $w \in W$;
  - **entangled states** are subsets that aren’t of this form.
I. Monoidal categories

In many theories, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to braided and symmetric monoidal categories.
I. Monoidal categories

In many theories, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to braided and symmetric monoidal categories.

**Definition 17.** A *braided monoidal category* is a monoidal category equipped with a natural isomorphism

$$A \otimes B \overset{\sigma_{A,B}}{\longrightarrow} B \otimes A$$

**I. Monoidal categories**

In many theories, the systems $A \otimes B$ and $B \otimes A$ can be considered essentially equivalent. Developing this idea gives rise to *braided* and *symmetric* monoidal categories.

**Definition 17.** A *braided monoidal category* is a monoidal category equipped with a natural isomorphism

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

satisfying the following *hexagon equations*:

\[
\begin{array}{c}
A \otimes (B \otimes C) \xrightarrow{\sigma_{A,B} \otimes C} (B \otimes C) \otimes A \\
\downarrow \alpha_{A,B,C}^{-1} \quad \downarrow \alpha_{B,C,A}^{-1} \\
(A \otimes B) \otimes C \quad B \otimes (C \otimes A) \\
\downarrow \sigma_{A,B} \otimes \text{id}_C \quad \downarrow \text{id}_B \otimes \sigma_{A,C} \\
(B \otimes A) \otimes C \quad B \otimes (A \otimes C) \\
\downarrow \alpha_{B,A,C} \\
(C \otimes A) \otimes B \quad A \otimes (B \otimes C) \\
\downarrow \text{id}_A \otimes \sigma_{B,C} \\
A \otimes (C \otimes B) \quad (A \otimes C) \otimes B \\
\downarrow \sigma_{A,C} \otimes \text{id}_B \\
\end{array}
\]
I. Monoidal categories

We include the braiding in our graphical notation like this:

\[
\begin{align*}
A \otimes B \xrightarrow{\sigma_{A,B}} & B \otimes A \\
B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} & A \otimes B
\end{align*}
\]
I. Monoidal categories

We include the braiding in our graphical notation like this:

\[
A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \quad \text{and} \quad B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B
\]

The strands of a braiding cross over each other, so the diagrams are not planar; they are inherently 3-dimensional.
I. Monoidal categories

We include the braiding in our graphical notation like this:

\[ A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A \]
\[ B \otimes A \xrightarrow{\sigma_{A,B}^{-1}} A \otimes B \]

The strands of a braiding cross over each other, so the diagrams are not planar; they are inherently 3-dimensional.

Invertibility takes the following graphical form:

\[ \begin{array}{c}
\text{=}
\end{array} \]
I. Monoidal categories

Naturality has the following graphical representation:
I. Monoidal categories

Naturality has the following graphical representation:

\[
\begin{align*}
\text{f} & \quad = \quad \text{g} \quad \text{f} \\
\text{f} & \quad \text{g} \quad = \quad \text{g} \quad \text{f}
\end{align*}
\]

The hexagon equations look like this:

\[
\begin{align*}
\text{f} & \quad = \quad \text{g} \\
\text{f} & \quad \text{g} \quad = \quad \text{g} \quad \text{f}
\end{align*}
\]

So braiding with a tensor product of two objects is the same as braiding with one then the other separately.
I. Monoidal categories

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

**Theorem 18.** A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.
I. Monoidal categories

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

**Theorem 18.** A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.

The coherence theorem is very powerful. For example, the following equations hold:

\[
\begin{align*}
\begin{array}{c}
\text{=}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{=}
\end{array}
\end{align*}
\]
I. Monoidal categories

Braided monoidal categories have a sound and complete graphical calculus, as established by the following theorem.

**Theorem 18.** A well-formed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the graphical language up to 3-dimensional isotopy.

The coherence theorem is very powerful. For example, the following equations hold:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0.5,0) to [out=90,in=90] (0.5,1);
  \draw (0.3,0) to [out=90,in=90] (0.3,1);
\end{tikzpicture}
\end{array}
& = \\
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
\end{tikzpicture}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
  \draw (0.5,0) to [out=90,in=90] (0.5,1);
  \draw (0.3,0) to [out=90,in=90] (0.3,1);
\end{tikzpicture}
\end{array}
& = \\
\begin{tikzpicture}
  \draw (0,0) -- (0,1);
  \draw (1,0) -- (1,1);
\end{tikzpicture}
\end{align*}
\]

The second equation is called the **Yang–Baxter equation**, which plays an important role in the mathematical theory of knots.
I. Monoidal categories

Let’s consider this structure for our example categories.
I. Monoidal categories

Let’s consider this structure for our example categories.

**Definition 19.** The monoidal categories **Hilb**, **Set** and **Rel** can all be equipped with a canonical braiding.

- In **Hilb**, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$. 
I. Monoidal categories

Let’s consider this structure for our example categories.

Definition 19. The monoidal categories $\text{Hilb}$, $\text{Set}$ and $\text{Rel}$ can all be equipped with a canonical braiding.

- In $\text{Hilb}$, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$.

- In $\text{Set}$, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \mapsto (b, a)$ for all $a \in A$ and $b \in B$. 
I. Monoidal categories

Let's consider this structure for our example categories.

**Definition 19.** The monoidal categories $\text{Hilb}$, $\text{Set}$ and $\text{Rel}$ can all be equipped with a canonical braiding.

- In $\text{Hilb}$, $H \otimes K \xrightarrow{\sigma_{H,K}} K \otimes H$ is the unique linear map extending $a \otimes b \mapsto b \otimes a$ for all $a \in H$ and $b \in K$.

- In $\text{Set}$, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \mapsto (b, a)$ for all $a \in A$ and $b \in B$.

- In $\text{Rel}$, $A \times B \xrightarrow{\sigma_{A,B}} B \times A$ is defined by $(a, b) \sim (b, a)$ for all $a \in A$ and $b \in B$. 
I. Monoidal categories

In Hilb, Rel and Set, the braidings satisfy an extra property.
I. Monoidal categories

In Hilb, Rel and Set, the braidings satisfy an extra property.

**Definition 20.** A braided monoidal category is *symmetric* when

\[ \sigma_{B,A} \circ \sigma_{A,B} = \text{id}_{A \otimes B} \]

for all objects \( A \) and \( B \), in which case we call \( \sigma \) the *symmetry*. 
I. Monoidal categories

In Hilb, Rel and Set, the braidings satisfy an extra property.

Definition 20. A braided monoidal category is symmetric when

$$\sigma_{B,A} \circ \sigma_{A,B} = \text{id}_{A \otimes B}$$

for all objects $A$ and $B$, in which case we call $\sigma$ the symmetry.

The symmetry condition has the following representation:

The strings can pass through each other, and knots can't be formed.
I. Monoidal categories

In Hilb, Rel and Set, the braidings satisfy an extra property.

**Definition 20.** A braided monoidal category is *symmetric* when

\[ \sigma_{B,A} \circ \sigma_{A,B} = \text{id}_{A \otimes B} \]

for all objects \( A \) and \( B \), in which case we call \( \sigma \) the *symmetry*. The symmetry condition has the following representation:

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{symmetry.png}} \\
\end{array} \]

The strings can pass through each other, and knots can’t be formed.

**Lemma 21.** In a symmetric monoidal category \( \sigma_{A,B} = \sigma_{B,A}^{-1} \), with the following graphical representation:

\[ \begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{lemma21.png}} \\
\end{array} \]
Part II

Higher categories
II. Higher categories

Definition 8.1. A 2-category $C$ consists of the following data:
II. Higher categories

Definition 8.1. A 2-category $\mathbf{C}$ consists of the following data:

- a collection $\text{Ob}(\mathbf{C})$ of objects;
II. Higher categories

Definition 8.1. A 2-category $\mathbf{C}$ consists of the following data:

- a collection $\text{Ob}(\mathbf{C})$ of objects;

- for any two objects $A, B$, a category $\mathbf{C}(A, B)$, with objects called 1-morphisms drawn as $A \xrightarrow{f} B$, and morphisms $\mu$ called 2-morphisms drawn as $f \xrightarrow{\mu} g$, or in full form as follows:

$$
\begin{array}{c}
& g \\
\mu & \Downarrow \\
B & \Rightarrow \\
& \mu \\
\downarrow & \\
A & \Rightarrow \\
f & \\
\end{array}
$$
II. Higher categories

- for 2-morphisms \( f \xrightarrow{\mu} g \) and \( g \xrightarrow{\nu} h \), an operation called *vertical composition* given by their composite as morphisms in \( \mathbf{C}(A, B) \):

\[
\begin{array}{c}
B \\
\mu
\end{array} \xleftarrow{\nu} \begin{array}{c}
A \\
\uparrow \\
\downarrow \\
\mu
\end{array} \xrightarrow{\nu} \begin{array}{c}
B \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\mu
\end{array}
\]
II. Higher categories

- for 2-morphisms \( f \xrightarrow{\mu} g \) and \( g \xrightarrow{\nu} h \), an operation called *vertical composition* given by their composite as morphisms in \( \mathcal{C}(A, B) \):

\[
\begin{array}{ccc}
& & h \\
& \uparrow{\nu} & \\
g & \leftarrow & A \\
& \uparrow{\mu} & \\
B & \leftarrow & \downarrow{f}
\end{array}
\]

- for any triple of objects \( A, B, C \) a *horizontal composition* functor:

\[
\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)
\]

\[
\begin{array}{ccc}
C & \xrightarrow{\nu \circ \mu} & A \\
& \leftarrow & \equiv \\
h \circ f & \leftarrow & \\
& \leftarrow & \equiv \\
& \downarrow{g} & \\
& \leftarrow & \equiv \\
& \leftarrow & \equiv \\
& \leftarrow & \equiv \\
\end{array}
\]
II. Higher categories

- for any object $A$, a 1-morphism $A \xrightarrow{id_A} A$ called the *identity 1-morphism*;
II. Higher categories

• for any object $A$, a 1-morphism $A \xrightarrow{\text{id}_A} A$ called the identity 1-morphism;

• a natural family of invertible 2-morphisms $f \circ \text{id}_A \xrightarrow{\rho_f} f$ and $\text{id}_B \circ f \xrightarrow{\lambda_f} f$ called the left and right unitors;
II. Higher categories

• for any object $A$, a 1-morphism $A \xrightarrow{id_A} A$ called the identity 1-morphism;

• a natural family of invertible 2-morphisms $f \circ id_A \xrightarrow{\rho_f} f$ and $id_B \circ f \xrightarrow{\lambda_f} f$ called the left and right unitors;

• a natural family of invertible 2-morphisms $(h \circ g) \circ f \xrightarrow{\alpha_{h,g,f}} h \circ (g \circ f)$ called the associators.
II. Higher categories

• for any object $A$, a 1-morphism $A \overset{\text{id}_A}{\longrightarrow} A$ called the identity 1-morphism;

• a natural family of invertible 2-morphisms $f \circ \text{id}_A \overset{\rho_f}{\longrightarrow} f$ and $\text{id}_B \circ f \overset{\lambda_f}{\longrightarrow} f$ called the left and right unitors;

• a natural family of invertible 2-morphisms $(h \circ g) \circ f \overset{\alpha_{h,g,f}}{\longrightarrow} h \circ (g \circ f)$ called the associators.

This structure is required to be coherent, meaning that any well-formed diagram built from the components of $\alpha$, $\lambda$, $\rho$ and their inverses under horizontal and vertical composition must commute.
II. Higher categories

• for any object $A$, a 1-morphism $A \xrightarrow{id_A} A$ called the identity 1-morphism;

• a natural family of invertible 2-morphisms $f \circ id_A \xrightarrow{\rho_f} f$ and $id_B \circ f \xrightarrow{\lambda_f} f$ called the left and right unitors;

• a natural family of invertible 2-morphisms $(h \circ g) \circ f \xrightarrow{\alpha_{h,g,f}} h \circ (g \circ f)$ called the associators.

This structure is required to be coherent, meaning that any well-formed diagram built from the components of $\alpha$, $\lambda$, $\rho$ and their inverses under horizontal and vertical composition must commute.

As for monoidal categories, coherence follows just from the triangle and pentagon equations.
II. Higher categories

- for any object $A$, a 1-morphism $A \xrightarrow{id_A} A$ called the identity 1-morphism;

- a natural family of invertible 2-morphisms $f \circ id_A \xrightarrow{\rho_f} f$ and $id_B \circ f \xrightarrow{\lambda_f} f$ called the left and right unitors;

- a natural family of invertible 2-morphisms $(h \circ g) \circ f \xrightarrow{\alpha_{h,g,f}} h \circ (g \circ f)$ called the associators.

This structure is required to be coherent, meaning that any well-formed diagram built from the components of $\alpha$, $\lambda$, $\rho$ and their inverses under horizontal and vertical composition must commute.

As for monoidal categories, coherence follows just from the triangle and pentagon equations.

A 2-category is strict just when every $\lambda_f$, $\rho_f$, $\alpha_{h,g,f}$ is an identity.
II. Higher categories

Theorem. A monoidal category is a 2-category with one object.
Theorem. A monoidal category is a 2-category with one object.

Proof. We sketch the correspondence with this table:

| Monoidal category | One-object 2-category |
II. Higher categories

**Theorem.** A monoidal category is a 2-category with one object.

**Proof.** We sketch the correspondence with this table:

<table>
<thead>
<tr>
<th>Monoidal category</th>
<th>One-object 2-category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects</td>
<td>1-morphisms</td>
</tr>
</tbody>
</table>
II. Higher categories

Theorem. A monoidal category is a 2-category with one object.

Proof. We sketch the correspondence with this table:

<table>
<thead>
<tr>
<th>Monoidal category</th>
<th>One-object 2-category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects</td>
<td>1-morphisms</td>
</tr>
<tr>
<td>Morphisms</td>
<td>2-morphisms</td>
</tr>
</tbody>
</table>
II. Higher categories

**Theorem.** A monoidal category is a 2-category with one object.

**Proof.** We sketch the correspondence with this table:

<table>
<thead>
<tr>
<th>Monoidal category</th>
<th>One-object 2-category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects</td>
<td>1-morphisms</td>
</tr>
<tr>
<td>Morphisms</td>
<td>2-morphisms</td>
</tr>
<tr>
<td>Composition</td>
<td>Vertical composition</td>
</tr>
</tbody>
</table>
II. Higher categories

Theorem. A monoidal category is a 2-category with one object.

Proof. We sketch the correspondence with this table:

<table>
<thead>
<tr>
<th>Monoidal category</th>
<th>One-object 2-category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects</td>
<td>1-morphisms</td>
</tr>
<tr>
<td>Morphisms</td>
<td>2-morphisms</td>
</tr>
<tr>
<td>Composition</td>
<td>Vertical composition</td>
</tr>
<tr>
<td>Tensor product</td>
<td>Horizontal composition</td>
</tr>
</tbody>
</table>
II. Higher categories

**Theorem.** A monoidal category is a 2-category with one object.

**Proof.** We sketch the correspondence with this table:

<table>
<thead>
<tr>
<th>Monoidal category</th>
<th>One-object 2-category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objects</td>
<td>1-morphisms</td>
</tr>
<tr>
<td>Morphisms</td>
<td>2-morphisms</td>
</tr>
<tr>
<td>Composition</td>
<td>Vertical composition</td>
</tr>
<tr>
<td>Tensor product</td>
<td>Horizontal composition</td>
</tr>
<tr>
<td>Unit object</td>
<td>Identity 1-morphism</td>
</tr>
</tbody>
</table>

The transformations $\alpha$, $\lambda$ and $\rho$ are the same for both structures.
II. Higher categories

Cat, the 2-category of categories, functors and natural transformations, is an important motivating example.
II. Higher categories

\textbf{Cat}, the 2-category of categories, functors and natural transformations, is an important motivating example.

\textbf{Definition}. The 2-category \textbf{Cat} is defined as follows:
II. Higher categories

Cat, the 2-category of categories, functors and natural transformations, is an important motivating example.

Definition. The 2-category Cat is defined as follows:
- objects are categories;
II. Higher categories

**Cat**, the 2-category of categories, functors and natural transformations, is an important motivating example.

**Definition.** The 2-category **Cat** is defined as follows:

- **objects** are categories;
- **1-morphisms** are functors;
II. Higher categories

Cat, the 2-category of categories, functors and natural transformations, is an important motivating example.

Definition. The 2-category Cat is defined as follows:

- **objects** are categories;
- **1-morphisms** are functors;
- **2-morphisms** are natural transformations;
**II. Higher categories**

\( \textbf{Cat} \), the 2-category of categories, functors and natural transformations, is an important motivating example.

**Definition.** The 2-category \( \textbf{Cat} \) is defined as follows:

- **objects** are categories;
- **1-morphisms** are functors;
- **2-morphisms** are natural transformations;
- **vertical composition** is componentwise composition of natural transformations, with \((\mu \cdot \nu)_A := \mu_A \circ \nu_A\);
II. Higher categories

**Cat**, the 2-category of categories, functors and natural transformations, is an important motivating example.

**Definition.** The 2-category **Cat** is defined as follows:

- **objects** are categories;
- **1-morphisms** are functors;
- **2-morphisms** are natural transformations;
- **vertical composition** is componentwise composition of natural transformations, with \((\mu \cdot \nu)_A := \mu_A \circ \nu_A\);
- **horizontal composition** is composition of functors.
II. Higher categories

In this more general graphical calculus, objects are represented by regions, 1-morphisms by vertically-oriented lines, and 2-morphisms by vertices:
II. Higher categories

In this more general graphical calculus, objects are represented by regions, 1-morphisms by vertically-oriented lines, and 2-morphisms by vertices:

The graphical calculus is the dual of the pasting diagram notation.
II. Higher categories

Horizontal and vertical composition is represented like this:

\[
\begin{align*}
C & \xrightarrow{j} B \\
B & \xrightarrow{\nu} \downarrow^\mu \quad \Downarrow^h \quad \downarrow^g \quad \Downarrow^f \quad \xrightarrow{\mu} \nu \circ \mu \\
A & \xleftarrow{\nu} B \\
A & \xleftarrow{\mu} \Downarrow^h \quad \Downarrow^g \quad \Downarrow^f \quad \xrightarrow{\mu} \nu \cdot \mu \\
\end{align*}
\]
II. Higher categories

When using the graphical notation, as for monoidal categories, the structures $\lambda$, $\rho$ and $\alpha$ are not depicted.
II. Higher categories

When using the graphical notation, as for monoidal categories, the structures $\lambda$, $\rho$ and $\alpha$ are not depicted.

There is also a correctness theorem, as we would expect.

**Theorem.** (Correctness of the graphical calculus for a 2-category) A well-formed equation between 2-morphisms in a 2-category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.
II. Higher categories

When using the graphical notation, as for monoidal categories, the structures $\lambda$, $\rho$ and $\alpha$ are not depicted.

There is also a correctness theorem, as we would expect.

**Theorem.** (Correctness of the graphical calculus for a 2-category) 
A well-formed equation between 2-morphisms in a 2-category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

If we have only a single object $A$, which we may as well denote by a region coloured white, then the graphical calculus is identical to that of a monoidal category.
II. Higher categories

We can use the graphical calculus to define equivalence.

**Definition.** In a 2-category, an *equivalence* is a pair of 1-morphisms \( A \xrightarrow{F} B \) and \( B \xrightarrow{G} A \), and 2-morphisms \( G \circ F \xrightarrow{\alpha} \text{id}_A \) and \( \text{id}_B \xrightarrow{\beta} F \circ G \):
II. Higher categories

We can use the graphical calculus to define equivalence.

**Definition.** In a 2-category, an *equivalence* is a pair of 1-morphisms $A \xrightarrow{F} B$ and $B \xrightarrow{G} A$, and 2-morphisms $G \circ F \xrightarrow{\alpha} \text{id}_A$ and $\text{id}_B \xrightarrow{\beta} F \circ G$:

They must satisfy the following equations:
II. Higher categories

**Definition.** In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a *right dual* $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \xrightarrow{\alpha} \text{id}_A$ and $\text{id}_B \xrightarrow{\beta} F \circ G$.
Definition. In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a right dual $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \xrightarrow{\alpha} \text{id}_A$ and $\text{id}_B \xrightarrow{\beta} F \circ G$ satisfying the snake equations:
II. Higher categories

**Definition.** In a 2-category, a 1-morphism $A \xrightarrow{L} B$ has a right dual $B \xrightarrow{R} A$ when there are 2-morphisms $G \circ F \xrightarrow{\alpha} \text{id}_A$ and $\text{id}_B \xrightarrow{\beta} F \circ G$ satisfying the snake equations:

\[
\begin{align*}
\alpha & = \quad = \\
\beta & = \quad = 
\end{align*}
\]

**Theorem.** In $\mathbf{Cat}$, a duality $F \dashv G$ is exactly an adjunction $F \dashv G$ between $F$ and $G$ as functors.
II. Higher categories

We now prove a nontrivial theorem relating equivalences and duals.
II. Higher categories

We now prove a nontrivial theorem relating equivalences and duals.

**Theorem.** In a 2-category, every equivalence gives rise to a dual equivalence.
II. Higher categories

We now prove a nontrivial theorem relating equivalences and duals.

**Theorem.** In a 2-category, every equivalence gives rise to a dual equivalence.

**Proof.** Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms $\alpha$ and $\beta$. Then we will build a new equivalence witnessed by $\alpha$ and $\beta'$, with $\beta'$ defined like this:
II. Higher categories

We now prove a nontrivial theorem relating equivalences and duals.

**Theorem.** In a 2-category, every equivalence gives rise to a dual equivalence.

**Proof.** Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms $\alpha$ and $\beta$. Then we will build a new equivalence witnessed by $\alpha$ and $\beta'$, with $\beta'$ defined like this:

Since $\alpha'$ is composed from invertible 2-morphisms it must itself be invertible, and so it is clear that $\alpha'$ and $\beta$ still give an equivalence.
II. Higher categories

We now demonstrate that the adjunction equations are satisfied.

The first adjunction equation takes following form:
II. Higher categories

The second is demonstrated as follows:
II. Higher categories

Since monoidal categories are just 2-categories with one object, we immediately have the following corollary.

**Corollary.** In a monoidal category, if \( A \otimes B \simeq B \otimes A \simeq I \), then \( A \dashv B \) and \( B \vdash A \).
II. Higher categories

Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!
II. Higher categories

Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!

Remember the 2d graphical calculus for 2-categories:

- objects correspond to planes;
- 1-morphisms correspond to wires;
- 2-morphisms correspond to vertices.
II. Higher categories

Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!

Remember the 2d graphical calculus for 2-categories:

- objects correspond to planes;
- 1-morphisms correspond to wires;
- 2-morphisms correspond to vertices.

For monoidal 2-categories, we simply extend this into 3d.
II. Higher categories

Monoidal 2-categories are hard to define. The definition is known, but it is long and complex. This is a big problem in the field!

Remember the 2d graphical calculus for 2-categories:

- objects correspond to planes;
- 1-morphisms correspond to wires;
- 2-morphisms correspond to vertices.

For monoidal 2-categories, we simply extend this into 3d.

**Tensor product.** Given 2-morphisms $f \xrightarrow{\mu} g$ and $h \xrightarrow{\nu} j$, the their tensor product 2-morphism $\mu \boxtimes \nu$ is given like this:
II. Higher categories

**Interchange.** Components can move freely in their separate layers. The order of 1-morphisms in separate sheets can be *interchanged*:
II. Higher categories

**Interchange.** Components can move freely in their separate layers. The order of 1-morphisms in separate sheets can be *interchanged*:

This process itself gives a 2-morphism, which is called an *interchanger*. These two interchangers are inverse to each other.
II. Higher categories

**Interchange.** Components can move freely in their separate layers. The order of 1-morphisms in separate sheets can be interchanged:

This process itself gives a 2-morphism, which is called an *interchanger*. These two interchangers are inverse to each other.

**Unit object.** A monoidal 2-category has a *unit object* $I$, represented by a ‘blank’ region.
II. Higher categories

Something interesting happens when we combine interchangers and the unit object. Consider the interchanger diagram, but with all 4 planar regions labelled by the unit object:
Something interesting happens when we combine interchangers and the unit object. Consider the interchanger diagram, but with all 4 planar regions labelled by the unit object:

We obtain the graphical representation of a *braiding*. 
II. Higher categories

Recall the following result which we saw earlier.

**Theorem.** A monoidal category is a 2-category with one object.
Recall the following result which we saw earlier.

**Theorem.** *A monoidal category is a 2-category with one object.*

We can now extend this as follows.

**Theorem.** *A braided monoidal category is a monoidal 2-category with one object.*
II. Higher categories

Recall the following result which we saw earlier.

**Theorem.** A monoidal category is a 2-category with one object.

We can now extend this as follows.

**Theorem.** A braided monoidal category is a monoidal 2-category with one object.

We can put this into context with notions of higher category.

**Theorem.** A monoidal 2-category is a 3-category with one object.
Recall the following result which we saw earlier.

**Theorem.** A monoidal category is a 2-category with one object.

We can now extend this as follows.

**Theorem.** A braided monoidal category is a monoidal 2-category with one object.

We can put this into context with notions of higher category.

**Theorem.** A monoidal 2-category is a 3-category with one object.

**Corollary.** A braided monoidal category is a 3-category with one object and one 1-morphism.
II. Higher categories

Recall the following result which we saw earlier.

**Theorem.** *A monoidal category is a 2-category with one object.*

We can now extend this as follows.

**Theorem.** *A braided monoidal category is a monoidal 2-category with one object.*

We can put this into context with notions of higher category.

**Theorem.** *A monoidal 2-category is a 3-category with one object.*

**Corollary.** *A braided monoidal category is a 3-category with one object and one 1-morphism.*

**Conjecture.** *A symmetric monoidal category is a 4-category with one object, one 1-morphism and one 2-morphism.*
Recall the following result which we saw earlier.

**Theorem.** A monoidal category is a 2-category with one object.

We can now extend this as follows.

**Theorem.** A braided monoidal category is a monoidal 2-category with one object.

We can put this into context with notions of higher category.

**Theorem.** A monoidal 2-category is a 3-category with one object.

**Corollary.** A braided monoidal category is a 3-category with one object and one 1-morphism.

**Conjecture.** A symmetric monoidal category is a 4-category with one object, one 1-morphism and one 2-morphism.

The emerging pattern here is called the *periodic table*, and was predicted by Baez and Dolan in 1995.
Part III

Coherence
Some monoidal categories have a particularly simple structure.

**Definition 22.** A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, $\lambda_A$ and $\rho_A$ are all identities.
III. Coherence

Some monoidal categories have a particularly simple structure.

**Definition 22.** A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, $\lambda_A$ and $\rho_A$ are all identities.

Later we will sketch the proof of the following theorem.

**Theorem 23.** Every monoidal category is monoidally equivalent to a strict monoidal category.
III. Coherence

Some monoidal categories have a particularly simple structure.

**Definition 22.** A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, $\lambda_A$ and $\rho_A$ are all identities.

Later we will sketch the proof of the following theorem.

**Theorem 23.** Every monoidal category is monoidally equivalent to a strict monoidal category.

This seems like a very useful thing. *But beware!* This is not enough:

\[(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad I \otimes A = A = A \otimes I\]

In particular, it does not ensure that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

The identity $(A \otimes B) \otimes C \xrightarrow{id} A \otimes (B \otimes C)$ might not be natural!
III. Coherence

Some monoidal categories have a particularly simple structure.

**Definition 22.** A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, $\lambda_A$ and $\rho_A$ are all identities.

Later we will sketch the proof of the following theorem.

**Theorem 23.** Every monoidal category is monoidally equivalent to a strict monoidal category.

This seems like a very useful thing. *But beware!* This is not enough:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad I \otimes A = A = A \otimes I$$

In particular, it does not ensure that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

The identity $(A \otimes B) \otimes C \xrightarrow{id} A \otimes (B \otimes C)$ might not be natural!

**Definition 24.** A category is *skeletal* when any two isomorphic objects are equal.
III. Coherence

Some monoidal categories have a particularly simple structure.

**Definition 22.** A monoidal category is *strict* if the morphisms $\alpha_{A,B,C}$, $\lambda_A$ and $\rho_A$ are all identities.

Later we will sketch the proof of the following theorem.

**Theorem 23.** Every monoidal category is monoidally equivalent to a strict monoidal category.

This seems like a very useful thing. *But beware!* This is not enough:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad I \otimes A = A = A \otimes I$$

In particular, it does not ensure that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$. The identity $(A \otimes B) \otimes C \xrightarrow{\text{id}} A \otimes (B \otimes C)$ might not be natural!

**Definition 24.** A category is *skeletal* when any two isomorphic objects are equal.

**Theorem.** Not every monoidal category is monoidally equivalent to a strict monoidal skeletal category.
III. Coherence

For the case of FHilb, everything works nicely.
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_C$ is defined as follows:
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_C$ is defined as follows:
- **objects** are natural numbers $0, 1, 2, \ldots$;
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_\mathbb{C}$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \rightarrow m$ are matrices of complex numbers with $m$ rows and $n$ columns;
III. Coherence

For the case of FHilb, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_\mathbb{C}$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \to m$ are matrices of complex numbers with $m$ rows and $n$ columns;
- **composition** is matrix multiplication;
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_C$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \to m$ are matrices of complex numbers with $m$ rows and $n$ columns;
- **composition** is matrix multiplication;
- **identities** $n \xrightarrow{\text{id}_n} n$ are identity matrices.
III. Coherence

For the case of \textbf{FHilb}, everything works nicely.

\textbf{Definition 25.} The skeletal category \( \text{Mat}_\mathbb{C} \) is defined as follows:

- \textbf{objects} are natural numbers 0, 1, 2, \ldots;
- \textbf{morphisms} \( n \to m \) are matrices of complex numbers with \( m \) rows and \( n \) columns;
- \textbf{composition} is matrix multiplication;
- \textbf{identities} \( n \xrightarrow{\text{id}_n} n \) are identity matrices.

\textbf{Definition 26.} The following structure makes \( \text{Mat}_\mathbb{C} \) strict monoidal:
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_\mathbb{C}$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \rightarrow m$ are matrices of complex numbers with $m$ rows and $n$ columns;
- **composition** is matrix multiplication;
- **identities** $n \xrightarrow{\text{id}_n} n$ are identity matrices.

**Definition 26.** The following structure makes $\text{Mat}_\mathbb{C}$ strict monoidal:

- **tensor product** is given on objects by $n \otimes m = nm$, and on morphisms by Kronecker product of matrices;
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_\mathbb{C}$ is defined as follows:

- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \rightarrow m$ are matrices of complex numbers with $m$ rows and $n$ columns;
- **composition** is matrix multiplication;
- **identities** $n \xrightarrow{\text{id}_n} n$ are identity matrices.

**Definition 26.** The following structure makes $\text{Mat}_\mathbb{C}$ strict monoidal:

- **tensor product** is given on objects by $n \otimes m = nm$, and on morphisms by Kronecker product of matrices;
- **the monoidal unit** is the natural number 1;
III. Coherence

For the case of $\text{FHilb}$, everything works nicely.

**Definition 25.** The skeletal category $\text{Mat}_C$ is defined as follows:
- **objects** are natural numbers $0, 1, 2, \ldots$;
- **morphisms** $n \to m$ are matrices of complex numbers with $m$ rows and $n$ columns;
- **composition** is matrix multiplication;
- **identities** $n \overset{\text{id}_n}{\to} n$ are identity matrices.

**Definition 26.** The following structure makes $\text{Mat}_C$ strict monoidal:
- **tensor product** is given on objects by $n \otimes m = nm$, and on morphisms by Kronecker product of matrices;
- the **monoidal unit** is the natural number $1$;
- **associators, left unitors and right unitors** are identity matrices.
III. Coherence

Definition 27. A monoidal functor $F: \mathbf{C} \to \mathbf{D}$ between monoidal categories is a functor equipped with natural isomorphisms

$$(F_2)_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

$F_0 : I \to F(I)$$
III. Coherence

Definition 27. A *monoidal functor* $F : C \to D$ between monoidal categories is a functor equipped with natural isomorphisms

$$(F_2)_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$$

$F_0 : I \to F(I)$

making the following diagrams commute:

$$
\begin{array}{ccc}
(F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\alpha_{F(A),F(B),F(C)}} & F(A) \otimes (F(B) \otimes F(C)) \\
(F_2)_{A,B} \otimes \text{id}_{F(C)} & \downarrow & \text{id}_{F(A)} \otimes (F_2)_{B,C} \\
F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\
(F_2)_{A \otimes B,C} & \downarrow & \downarrow (F_2)_{A,B \otimes C} \\
F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{A,B,C})} & F(A \otimes (B \otimes C))
\end{array}
$$
III. Coherence

Definition 27. A monoidal functor $F: \mathbf{C} \to \mathbf{D}$ between monoidal categories is a functor equipped with natural isomorphisms

$$(F_2)_{A,B}: F(A) \otimes F(B) \to F(A \otimes B)$$

$$F_0: I \to F(I)$$

making the following diagrams commute:

![Diagram of coherence conditions](image-url)
III. Coherence

Definition 28. A monoidal equivalence is a monoidal functor that is an equivalence as a functor.
III. Coherence

**Definition 28.** A *monoidal equivalence* is a monoidal functor that is an equivalence as a functor.

**Theorem.** There is a monoidal equivalence $R: \mathbf{Mat}_C \to \mathbf{FHilb}$. 

Definition 28. A monoidal equivalence is a monoidal functor that is an equivalence as a functor.

Theorem. There is a monoidal equivalence $R : \text{Mat}_\mathbb{C} \to \text{FHilb}$.

Proof. We define $R$ like this:

$$R(n) := \mathbb{C}^n$$

$$R(n \xrightarrow{f} m) := f \text{ as a linear map}$$
III. Coherence

**Definition 28.** A monoidal equivalence is a monoidal functor that is an equivalence as a functor.

**Theorem.** There is a monoidal equivalence $R : \text{Mat}_\mathbb{C} \to \text{FHilb}$. 

**Proof.** We define $R$ like this:

\[
R(n) := \mathbb{C}^n \\
R(n \xrightarrow{f} m) := f \text{ as a linear map} \\
(R_2)_{m,n} : |i\rangle \otimes |j\rangle \mapsto |ni + j\rangle \\
R_0 : 1 \mapsto 1
\]

This is full, faithful and essentially surjective, and satisfies the monoidal functor conditions.
III. Coherence

We now prove the strictification theorem.

**Theorem 29.** Every monoidal category is monoidally equivalent to a strict monoidal category.
III. Coherence

We now prove the strictification theorem.

**Theorem 29.** Every monoidal category is monoidally equivalent to a strict monoidal category.

**Proof sketch.** Let $\mathcal{C}$ be a monoidal category, and define $\mathcal{D}$ like this:

- an object is $F : \mathcal{C} \to \mathcal{C}$ equipped with a natural isomorphism

$$F(A) \otimes B \overset{\gamma_{A,B}}{\longrightarrow} F(A \otimes B);$$
III. Coherence

We now prove the strictification theorem.

**Theorem 29.** Every monoidal category is monoidally equivalent to a strict monoidal category.

**Proof sketch.** Let \( \mathcal{C} \) be a monoidal category, and define \( \mathcal{D} \) like this:

- an object is \( F : \mathcal{C} \to \mathcal{C} \) equipped with a natural isomorphism
  \[
  F(A) \otimes B \xrightarrow{\gamma_{A,B}} F(A \otimes B);
  \]

- a morphism \( (F, \gamma) \to (F', \gamma') \) is \( \theta : F \Rightarrow F' \) such that:
  \[
  \begin{array}{ccc}
  F(A) \otimes B & \xrightarrow{\gamma_{A,B}} & F(A \otimes B) \\
  \downarrow \theta_A \otimes \text{id}_B & & \downarrow \theta_{A \otimes B} \\
  F'(A) \otimes B & \xrightarrow{\gamma'_{A,B}} & F'(A \otimes B)
  \end{array}
  \]
III. Coherence

Proof sketch (continued).

• the tensor product is $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where $\delta$ is

$$F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).$$
III. Coherence

Proof sketch (continued).

- the tensor product is $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where $\delta$ is
  
  
  \[
  F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).
  \]

We can then calculate these products:

\[
((F, \gamma) \otimes (F', \gamma')) \otimes (F'', \gamma'') \quad (F, \gamma) \otimes ((F', \gamma') \otimes (F'', \gamma''))
\]
III. Coherence

Proof sketch (continued).

• the tensor product is \((F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)\), where \(\delta\) is

\[
F(F'(A)) \otimes B \overset{\gamma_{F'(A),B}}{\longrightarrow} F(F'(A) \otimes B) \overset{F(\gamma'_{A,B})}{\longrightarrow} F(F'(A \otimes B)).
\]

We can then calculate these products:

\[
((F, \gamma) \otimes (F', \gamma')) \otimes (F'', \gamma'') = (F, \gamma) \otimes ((F', \gamma') \otimes (F'', \gamma'')).
\]

They are equal, and indeed the category is strict monoidal.
III. Coherence

Proof sketch (continued).

• the tensor product is \((F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)\), where \(\delta\) is

\[
F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).
\]

We can then calculate these products:

\[
((F, \gamma) \otimes (F', \gamma')) \otimes (F'', \gamma'') = (F, \gamma) \otimes ((F', \gamma') \otimes (F'', \gamma'')).
\]

They are equal, and indeed the category is strict monoidal.

Now build a monoidal functor \(L : \textbf{C} \to \textbf{D}\) in the following way:

\[
L(A) := (A \otimes -, \alpha_{A,-,-})
\]

You can show that \(L\) is full and faithful.
III. Coherence

Proof sketch (continued).

• the tensor product is $(F, \gamma) \otimes (F', \gamma') := (F \circ F', \delta)$, where $\delta$ is

$$F(F'(A)) \otimes B \xrightarrow{\gamma_{F'(A),B}} F(F'(A) \otimes B) \xrightarrow{F(\gamma'_{A,B})} F(F'(A \otimes B)).$$

We can then calculate these products:

$$(F, \gamma) \otimes (F', \gamma') \otimes (F'', \gamma'') = (F, \gamma) \otimes ((F', \gamma') \otimes (F'', \gamma''))$$

They are equal, and indeed the category is strict monoidal.

Now build a monoidal functor $L: \mathbf{C} \rightarrow \mathbf{D}$ in the following way:

$$L(A) := (A \otimes -, \alpha_{A,-,-})$$

You can show that $L$ is full and faithful.

Finally, restrict $\mathbf{D}$ to the strict monoidal subcategory containing objects isomorphic to those in the image of $L$. Then $L$ is a monoidal equivalence of $\mathbf{C}$ with a strict monoidal category.
The final topic in this chapter is *coherence*: any well-formed equation built from $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes$ and $\circ$ holds.
III. Coherence

The final topic in this chapter is coherence: any well-formed equation built from $\alpha$, $\alpha^{-1}$, $\lambda$, $\lambda^{-1}$, $\rho$, $\rho^{-1}$, id, $\otimes$ and $\circ$ holds.

An equation is well-formed when it does not make use of any ‘accidental equalities’ of objects. For example, suppose that $(A \otimes A) \otimes A = A \otimes (A \otimes A) = A$. Then

$$\alpha_{A,A,A} = \text{id}_A$$

is not well-formed.
The final topic in this chapter is *coherence*: any well-formed equation built from $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes$ and $\circ$ holds.

An equation is *well-formed* when it does not make use of any ‘accidental equalities’ of objects. For example, suppose that $(A \otimes A) \otimes A = A \otimes (A \otimes A) = A$. Then

$$\alpha_{A,A,A} = \text{id}_A$$

is not well-formed.

To make this precise, let a *bracketing* be a fixed way to bracket a list of objects of a given length, including empty brackets. For example, we could define the following bracketings $v, w$:

$$v(A, B, C, D) = ((A \otimes B) \otimes ()) \otimes (C \otimes D)$$
$$w(A, B, C, D) = ((()) \otimes (A \otimes (B \otimes C))) \otimes ((()) \otimes ((()) \otimes D)))$$

Then we can consider transformations of bracketings $\theta, \theta' : v \Rightarrow \mu$. 

III. Coherence
III. Coherence

We now give a proof of the coherence theorem.

**Theorem 30.** Let $v, w$ be bracketings; then any two transformations $\theta, \theta': v \Rightarrow w$ built from $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes, \text{and } \circ$ are equal.
III. Coherence

We now give a proof of the coherence theorem.

**Theorem 30.** Let $\nu, \omega$ be bracketings; then any two transformations $\theta, \theta': \nu \Rightarrow \omega$ built from $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes, \circ$ are equal.

**Proof.** We can define a canonical morphism

$$\nu(L(A), \ldots, L(Z)) \xrightarrow{L_{\nu}} L(\nu(A, \ldots, Z))$$

using the fact that $L$ is a monoidal functor, and similarly for $\omega$.
III. Coherence

We now give a proof of the coherence theorem.

**Theorem 30.** Let \( v, w \) be bracketings; then any two transformations \( \theta, \theta': v \Rightarrow w \) built from \( \alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes, \) and \( \circ \) are equal.

**Proof.** We can define a canonical morphism

\[
v(L(A), \ldots, L(Z)) \xrightarrow{L_v} L(v(A, \ldots, Z))
\]

using the fact that \( L \) is a monoidal functor, and similarly for \( w \).

Then the following diagram commutes, for both \( \theta \) and \( \theta' \):

\[
\begin{array}{c}
\begin{array}{c}
\nu(L(A), \ldots, L(Z)) \\
\downarrow L_{v}^{-1}
\end{array}
\xrightarrow{\theta(L(A), \ldots, L(Z))} \\
\begin{array}{c}
L(\nu(A, \ldots, Z)) \\
\downarrow L_w
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\nu(L(A), \ldots, L(Z)) \\
\downarrow L_{v}^{-1}
\end{array}
\xrightarrow{\theta(L(A), \ldots, L(Z))} \\
\begin{array}{c}
L(\nu(A, \ldots, Z)) \\
\downarrow L_w
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\nu(L(A), \ldots, L(Z)) \\
\downarrow L_{v}^{-1}
\end{array}
\xrightarrow{\theta(L(A), \ldots, L(Z))} \\
\begin{array}{c}
L(\nu(A, \ldots, Z)) \\
\downarrow L_w
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\nu(L(A), \ldots, L(Z)) \\
\downarrow L_{v}^{-1}
\end{array}
\xrightarrow{\theta(L(A), \ldots, L(Z))} \\
\begin{array}{c}
L(\nu(A, \ldots, Z)) \\
\downarrow L_w
\end{array}
\end{array}
\]
III. Coherence

We now give a proof of the coherence theorem.

**Theorem 30.** Let $v, w$ be bracketings; then any two transformations $\theta, \theta' : v \Rightarrow w$ built from $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}, \text{id}, \otimes, \text{and} \circ$ are equal.

**Proof.** We can define a canonical morphism

$$v(L(A), \ldots, L(Z)) \xrightarrow{L_v} L(v(A, \ldots, Z))$$

using the fact that $L$ is a monoidal functor, and similarly for $w$. Then the following diagram commutes, for both $\theta$ and $\theta'$:

$$
\begin{array}{ccc}
L_v^{-1} & \theta(L(A), \ldots, L(Z)) & L_w \\
\downarrow & \downarrow & \\
L(v(A, \ldots, Z)) & \xrightarrow{\theta(A, \ldots, Z)} & L(w(A, \ldots, Z))
\end{array}
$$

But $\theta(L(A), \ldots, L(Z)) = \theta'(L(A), \ldots, L(Z)) = \text{id}!$ So $L(\theta(A, \ldots, Z)) = L(\theta'(A, \ldots, Z))$, and hence $\theta(A, \ldots, Z) = \theta'(A, \ldots, Z)$, since $L$ is faithful.
Part IV

Duals in monoidal categories
IV. Duals in monoidal categories

Dual objects have two basic interpretations:
IV. Duals in monoidal categories

Dual objects have two basic interpretations:
  • Topologically, they allow wires to bend
IV. Duals in monoidal categories

Dual objects have two basic interpretations:

- Topologically, they allow wires to bend
- Quantum mechanically, they model full-rank entangled states
IV. Duals in monoidal categories

Dual objects have two basic interpretations:

- Topologically, they allow wires to bend
- Quantum mechanically, they model full-rank entangled states

**Definition 31.** An object $L$ is *left-dual* to an object $R$, and $R$ is *right-dual* to $L$, written $L \dashv R$, when there is a unit morphism $\eta: I \rightarrow R \otimes L$ and a counit morphism $L \otimes R \rightarrow I$ such that:
IV. Duals in monoidal categories

Dual objects have two basic interpretations:

- Topologically, they allow wires to bend
- Quantum mechanically, they model full-rank entangled states

**Definition 31.** An object $L$ is *left-dual* to an object $R$, and $R$ is *right-dual* to $L$, written $L \vdash R$, when there is a unit morphism $I \xrightarrow{\eta} R \otimes L$ and a counit morphism $L \otimes R \xrightarrow{\varepsilon} I$ such that:

$$
\begin{align*}
L & \xrightarrow{\rho_L^{-1}} L \otimes I \xrightarrow{id_L \otimes \eta} L \otimes (R \otimes L) \\
L & \xleftarrow{\lambda_L} I \otimes L \xleftarrow{\varepsilon \otimes id_L} (L \otimes R) \otimes L \\
R & \xrightarrow{\rho_R} R \otimes I \xleftarrow{id_R \otimes \varepsilon} R \otimes (L \otimes R) \\
R & \xleftarrow{\lambda_R^{-1}} I \otimes R \xrightarrow{\eta \otimes id_R} (R \otimes L) \otimes R
\end{align*}
$$

$$
\begin{align*}
\alpha_{L,R,L}^{-1} & \downarrow \\
\alpha_{L,R,L} & \downarrow \\
\alpha_{R,L,R}^{-1} & \downarrow
\end{align*}
$$
IV. Duals in monoidal categories

We draw an object $L$ as a wire with an upward-pointing arrow, and a right dual $R$ as a wire with a downward-pointing arrow.
IV. Duals in monoidal categories

We draw an object $L$ as a wire with an upward-pointing arrow, and a right dual $R$ as a wire with a downward-pointing arrow.

![Diagram of $L$ and $R$]

The unit $I \xrightarrow{\eta} R \otimes L$ and counit $L \otimes R \xrightarrow{\varepsilon} I$ are drawn as bent wires:

![Diagram of bent wires]

This notation is chosen because of the attractive form it gives to the duality equations:

![Diagram of snake equations]

They are also called the *snake equations*. 
IV. Duals in monoidal categories

The monoidal category $\mathbf{FHilb}$ has all duals. Every finite-dimensional Hilbert space $H$ is both right dual and left dual to its dual Hilbert space $H^*$, in a canonical way.

Of course, this is the origin of the terminology.
IV. Duals in monoidal categories

The monoidal category \textbf{FHilb} has all duals. Every finite-dimensional Hilbert space $H$ is both right dual and left dual to its dual Hilbert space $H^*$, in a canonical way.

Of course, this is the origin of the terminology.

The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is defined like this:

$$\varepsilon: \langle \phi \rangle \otimes \langle \psi \rangle \mapsto \langle \psi \vert \phi \rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $\vert i \rangle$:

$$\eta: 1 \mapsto \sum_i \langle i \vert \otimes \vert i \rangle$$
IV. Duals in monoidal categories

The monoidal category $\textbf{FHilb}$ has all duals. Every finite-dimensional Hilbert space $H$ is both right dual and left dual to its dual Hilbert space $H^*$, in a canonical way.

Of course, this is the origin of the terminology.

The counit $H \otimes H^* \xrightarrow{\varepsilon} \mathbb{C}$ is defined like this:

$$\varepsilon: \langle \phi \rangle \otimes \langle \psi \rangle \mapsto \langle \psi | \phi \rangle$$

The unit $\mathbb{C} \xrightarrow{\eta} H^* \otimes H$ is defined like so, for any orthonormal basis $|i\rangle$:

$$\eta: 1 \mapsto \sum_i \langle i | \otimes | i \rangle$$

These definitions sit together rather oddly: $\eta$ seems basis-dependent, while $\varepsilon$ is clearly not.

In fact the same value of $\eta$ is obtained whatever orthonormal basis is used, as we will see below.

Infinite-dimensional spaces do not have duals.
In Rel, every object is its own dual, even sets of infinite cardinality. The unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ can be defined like this:

- $\sim_\eta (s, s)$ for all $s \in S$
- $(s, s) \sim_\varepsilon \bullet$ for all $s \in S$
IV. Duals in monoidal categories

In **Rel**, every object is its own dual, even sets of infinite cardinality. The unit $1 \xrightarrow{\eta} S \times S$ and counit $S \times S \xrightarrow{\varepsilon} 1$ can be defined like this:

$$
\bullet \sim_{\eta} (s, s) \text{ for all } s \in S \\
(s, s) \sim_{\varepsilon} \bullet \text{ for all } s \in S
$$

In **Mat**$_{\mathbb{C}}$, every object $n$ is its own dual, with a canonical choice of $\eta$ and $\varepsilon$ given as follows:

$$
\eta : 1 \mapsto \sum_{i} |i\rangle \otimes |i\rangle \quad \varepsilon : |i\rangle \otimes |j\rangle \mapsto \delta_{ij} 1
$$
IV. Duals in monoidal categories

The category $\text{Set}$ only has duals for sets of size 1. Let's see why.

**Definition 32.** In a monoidal category with dualities $A \dashv A^*$ and $B \dashv B^*$, given a morphism $A \xrightarrow{f} B$, we define its *name* $I \xleftarrow{\downharpoonup} A^* \otimes B$ and *coname* $A \otimes B^* \xrightarrow{\downharpoonright} I$ as the following morphisms:

\[
\begin{align*}
A^* & \xrightarrow{f^\ast} B \\
\end{align*}
\]

Morphisms can be recovered from their names or conames:

\[
\begin{align*}
B & \xrightarrow{f} B \\
\end{align*}
\]

In $\text{Set}$, 1 is terminal, and so all conames $A \otimes B^* \xrightarrow{\downharpoonright} 1$ must be equal. If $\text{Set}$ had duals this would imply all functions $A \rightarrow B$ were equal.
IV. Duals in monoidal categories

We first show duals are well-defined up to canonical isomorphism.

**Lemma 33.** In a monoidal category with \( L \dashv R \), then \( L \dashv R' \) if and only if \( R \simeq R' \). Similarly, if \( L \dashv R \), then \( L' \dashv R \) if and only if \( L \simeq L' \).
IV. Duals in monoidal categories

We first show duals are well-defined up to canonical isomorphism.

**Lemma 33.** In a monoidal category with $L \dashv R$, then $L \dashv R'$ if and only if $R \simeq R'$. Similarly, if $L \vdash R$, then $L' \vdash R$ if and only if $L \simeq L'$.

**Proof.** If $L \dashv R$ and $L \dashv R'$, define maps $R \to R'$ and $R' \to R$ as follows:

The snake equations imply that these are inverse.
We first show duals are well-defined up to canonical isomorphism.

**Lemma 33.** In a monoidal category with \( L \dashv R \), then \( L \dashv R' \) if and only if \( R \simeq R' \). Similarly, if \( L \vdash R \), then \( L' \vdash R \) if and only if \( L \simeq L' \).

**Proof.** If \( L \vdash R \) and \( L \vdash R' \), define maps \( R \to R' \) and \( R' \to R \) as follows:

\[
\begin{align*}
R' & \xrightarrow{L} R \\
R & \xrightarrow{L} R'
\end{align*}
\]

The snake equations imply that these are inverse. Conversely, if \( L \vdash R \) and \( R \xrightarrow{f} R' \) is invertible, we can construct a duality \( L \vdash R' \):

\[
\begin{align*}
L & \xrightarrow{\overline{f}} R' \\
R' & \xrightarrow{f} L \\
R & \xrightarrow{\overline{f}} R'
\end{align*}
\]
IV. Duals in monoidal categories

Given a duality, the unit determines the counit, and vice-versa.
IV. Duals in monoidal categories

Given a duality, the unit determines the counit, and vice-versa.

**Lemma 34.** In a monoidal category, if \((L, R, \eta, \varepsilon)\) and \((L, R, \eta, \varepsilon')\) both exhibit a duality, then \(\varepsilon = \varepsilon'\). Similarly, if \((L, R, \eta, \varepsilon)\) and \((L, R, \eta', \varepsilon)\) both exhibit a duality, then \(\eta = \eta'\).
Given a duality, the unit determines the counit, and vice-versa.

**Lemma 34.** In a monoidal category, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta, \varepsilon')$ both exhibit a duality, then $\varepsilon = \varepsilon'$. Similarly, if $(L, R, \eta, \varepsilon)$ and $(L, R, \eta', \varepsilon)$ both exhibit a duality, then $\eta = \eta'$.

**Proof.** For the first case, we use the following graphical argument.

\[
\begin{array}{ccc}
\varepsilon & = & \varepsilon' \\
& \Downarrow & \Downarrow \\
& \varepsilon' & = \varepsilon'
\end{array}
\]

The second case is similar.
IV. Duals in monoidal categories

The following lemma shows that dual objects interact well with the monoidal structure.
IV. Duals in monoidal categories

The following lemma shows that dual objects interact well with the monoidal structure.

**Lemma 35.** In a monoidal category, $I \dashv I$.

**Proof.** Taking $\eta = \lambda_I^{-1} : I \to I \otimes I$ and $\varepsilon = \lambda_I : I \otimes I \to I$ shows that $I \dashv I$. The snake equations follow from the coherence theorem. □
IV. Duals in monoidal categories

The following lemma shows that dual objects interact well with the monoidal structure.

**Lemma 35.** In a monoidal category, $I \dashv I$.

**Proof.** Taking $\eta = \lambda_I^{-1} : I \to I \otimes I$ and $\varepsilon = \lambda_I : I \otimes I \to I$ shows that $I \dashv I$. The snake equations follow from the coherence theorem.

**Lemma 36.** In a monoidal category, $L \dashv R$, $L' \dashv R' \Rightarrow L \otimes L' \dashv R' \otimes R$.

**Proof.** Suppose that $L \dashv R$ and $L' \dashv R'$. We make the new unit and counit maps from the old ones, and compute as follows:

The other snake equation follows similarly.
IV. Duals in monoidal categories

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

**Lemma 37.** In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$. 
IV. Duals in monoidal categories

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

**Lemma 37.** In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

**Proof.** Construct a new duality as follows:

- $I \xrightarrow{\eta'} L \otimes R$
- $R \otimes L \xrightarrow{\varepsilon'} I$
IV. Duals in monoidal categories

If the monoidal category has a braiding then a duality $L \dashv R$ gives rise to a duality $R \dashv L$, as the next lemma investigates.

**Lemma 37.** In a braided monoidal category, $L \dashv R \Rightarrow R \dashv L$.

**Proof.** Construct a new duality as follows:

\[ I \xrightarrow{\eta'} L \otimes R \]
\[ R \otimes L \xrightarrow{\varepsilon'} I \]

We can then test the snake equations:

\[ \begin{array}{c}
\end{array} \]

The other snake equation can be proved in a similar way.
Next we will prove some nice theorems showing the relationship between duals and monoidal functors.

To understand them, we will need to develop a graphical calculus for monoidal functors.
IV. Duals in monoidal categories

Next we will prove some nice theorems showing the relationship between duals and monoidal functors.

To understand them, we will need to develop a graphical calculus for monoidal functors.

We depict a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ and the isomorphisms $(F_2)_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$ and $F_0 : I \to F(I)$ like this:
IV. Duals in monoidal categories

Naturality means that morphisms can pass through the gaps:

\[ f \circ g = g \circ f \]
IV. Duals in monoidal categories

Naturality means that morphisms can pass through the gaps:

\[ f \circ g = f \circ g \]

The coherence equations look like this:

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array} \]

They have a nice topological flavour.
Let’s prove our first theorem using these techniques.

**Theorem 38.** Monoidal functors preserve duals.
Let’s prove our first theorem using these techniques.

**Theorem 38.** Monoidal functors preserve duals.

**Proof.** If we apply our monoidal functor to the unit and counit, we can show that the duality equations are still satisfied:

\[
\begin{align*}
\text{\includegraphics[width=0.6\textwidth]{duality_equation.png}}
\end{align*}
\]

The other duality equation can be proved in a similar way.
IV. Duals in monoidal categories

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$ and a natural transformation $\mu : F \Rightarrow G$, we can denote it like this:

If $\mathbf{C}, \mathbf{D}, F, G$ and $\mu$ are monoidal, then we have following extra properties:
IV. Duals in monoidal categories

**Theorem 39.** Let $\mu : F \Rightarrow G$ be a monoidal natural transformation. If $A \in \text{Ob}(\mathcal{C})$ has a left or a right dual, $F(A) \xrightarrow{\mu_A} G(A)$ is invertible.

**Proof.** Choose $A = L$ with $L \dashv R$ in $\mathcal{C}$. Then we perform the following computation:

\[
\mu_L \mu_R = \mu_{L \otimes R} = \mu_I
\]

The rest of the proof uses similar techniques.
IV. Duals in monoidal categories

Choosing duals for objects extends functorially to morphisms.
Choosing duals for objects extends functorially to morphisms.

**Definition 40.** For a morphism $A \xrightarrow{f} B$ and chosen dualities $A \dashv A^*$, $B \dashv B^*$, the *right dual* $B^* \xrightarrow{f^*} A^*$ is defined in the following way:

We represent this graphically by rotating the box representing $f$, as shown in the third image above.
The dual can ‘slide’ along the unit and counit.

**Lemma 41.** In a monoidal category with chosen dualities $A \dashv A^*$ and $B \dashv B^*$, the following equations hold for all morphisms $A \xrightarrow{f} B$:

$$f = f$$

$$\Rightarrow$$

$$f = f$$

**Proof.** Let’s write it out on the board. \[\square\]
Lemma 42. If a monoidal category has assigned right duals, the right-duals construction \((-)^*\) defines a functor.

**Proof.** Let \(A \xrightarrow{f} B\) and \(B \xrightarrow{g} C\). Then we perform the following calculation:

\[
(g \circ f)^* = f^* \circ g^* = f^* \circ g^* \quad \text{Similarly, } (\text{id}_A)^* = \text{id}_A^* \text{ follows from the snake equations.}
\]
Example 43. Let’s see how the right duals functor acts for our example categories, with chosen right duals as given earlier.
Example 43. Let’s see how the right duals functor acts for our example categories, with chosen right duals as given earlier.

- In $\text{FVect}$ and $\text{FHilb}$, the right dual of a morphism $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of $W^*$. 
Example 43. Let’s see how the right duals functor acts for our example categories, with chosen right duals as given earlier.

- In $\text{FVect}$ and $\text{FHilb}$, the right dual of a morphism $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{\epsilon} \mathbb{C}$ is an arbitrary element of $W^*$.

- In $\text{Mat}_\mathbb{C}$, the dual of a matrix is its transpose.
Example 43. Let’s see how the right duals functor acts for our example categories, with chosen right duals as given earlier.

- In $\mathbf{FVect}$ and $\mathbf{FHilb}$, the right dual of a morphism $V \xrightarrow{f} W$ is $W^* \xrightarrow{f^*} V^*$, acting as $f^*(e) := e \circ f$, where $W \xrightarrow{e} \mathbb{C}$ is an arbitrary element of $W^*$.

- In $\mathbf{Mat}_\mathbb{C}$, the dual of a matrix is its transpose.

- In $\mathbf{Rel}$, the dual of a relation is its converse. So the right duals functor and the dagger functor have the same action: $R^* = R^\dagger$ for all relations $R$. 
Lemma 44. For a monoidal category with chosen right duals for objects, the double duals functor $(-)^{**} : C \to C$ is monoidal.

Proof. The isomorphism $A^{**} \otimes B^{**} \simeq (A \otimes B)^{**}$ looks like this:

\[
\begin{array}{ccc}
A^{**} & \xrightarrow{\eta(A \otimes B)^*} & (A \otimes B)^* \\
\downarrow & \swarrow & \downarrow \\
\varepsilon_{A \otimes B} & & (A \otimes B)^{**} \\
\end{array}
\]

Showing this satisfies the monoidal functor axioms is a monster! \(\square\)
IV. Duals in monoidal categories

Definition 45. A monoidal category with right duals is *pivotal* when it is equipped with a monoidal natural transformation $A \xrightarrow{p_A} A^{**}$. 
IV. Duals in monoidal categories

Definition 45. A monoidal category with right duals is *pivotal* when it is equipped with a monoidal natural transformation $A \xrightarrow{pA} A^{**}$. By Theorem 39, this will necessarily be invertible.
IV. Duals in monoidal categories

Definition 45. A monoidal category with right duals is pivotal when it is equipped with a monoidal natural transformation $A \xrightarrow{p_A} A^{**}$. By Theorem 39, this will necessarily be invertible.

In a pivotal category, we extend the graphical calculus:

\[
\begin{align*}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1); \draw[->] (0,1) -- (1,0);
\end{tikzpicture}
\quad := \quad
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1); \draw[->] (0,1) -- (1,0);
\node at (0.5,0.5) {$\pi_A$};
\end{tikzpicture}
\quad \begin{tikzpicture}
\draw[->] (0,0) -- (1,1); \draw[->] (0,1) -- (1,0); \draw[->] (1,0) -- (0,1);
\node at (0.5,0.5) {$\pi_A^{-1}$};
\end{tikzpicture}
\end{align*}
\]

We can use this to rotate boxes arbitrarily.
### IV. Duals in monoidal categories

**Definition 45.** A monoidal category with right duals is *pivotal* when it is equipped with a monoidal natural transformation $A \xrightarrow{\mathcal{P}_A} A^{**}$.

By Theorem 39, this will necessarily be invertible.

In a pivotal category, we extend the graphical calculus:

![Graphical representation of pivotal category](image)

We can use this to rotate boxes arbitrarily.

**Lemma.** In a pivotal category, the following equations hold for all morphisms $A \xrightarrow{f} B$:

![Equations for pivotal category](image)

**Proof.** Let’s write it out on the board.
IV. Duals in monoidal categories

We can formalize this as follows.

**Theorem 46.** A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.
IV. Duals in monoidal categories

We can formalize this as follows.

**Theorem 46.** A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:
IV. Duals in monoidal categories

We can formalize this as follows.

**Theorem 46.** A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:
IV. Duals in monoidal categories

We can formalize this as follows.

**Theorem 46.** A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:
We can formalize this as follows.

**Theorem 46.** A well-formed equation between morphisms in a pivotal category follows from the axioms if and only if it holds in the graphical language up to planar oriented isotopy.

The new feature is the word *oriented*. The wires of our diagram have arrows, and an isotopy must preserve them:
Definition 47. A braided monoidal category is balanced when it is equipped with a natural isomorphism $\theta_A : A \to A$ called a twist, satisfying the following equations:

The second equation here says $\theta_I = \text{id}_I$. 
Definition 47. A braided monoidal category is balanced when it is equipped with a natural isomorphism \( \theta_A : A \to A \) called a twist, satisfying the following equations:

\[
\theta_{A \otimes B} = \theta_A \otimes \theta_B = \theta_I
\]

The second equation here says \( \theta_I = \text{id}_I \).

These equations look strange—we will see later what they mean!
IV. Duals in monoidal categories

**Theorem 48.** For a braided monoidal category with duals, a pivotal structure uniquely induces a twist structure, and vice versa.
Theorem 48. For a braided monoidal category with duals, a pivotal structure uniquely induces a twist structure, and vice versa.

Proof. Suppose we have a twist structure $\theta_A : A \to A$. Then define a pivotal structure as follows:

$$\pi_A : A^{**} \to A$$

We must verify that it is a monoidal natural transformation, and that it is natural.
IV. Duals in monoidal categories

For the monoidal property, perform the following calculation:

\[
\pi_{A \otimes B} = \theta^{-1}_{A \otimes B} = \theta^{-1}_A \theta^{-1}_B \text{ iso } = \theta^{-1}_B \theta^{-1}_A \text{ iso } = \pi_A \otimes \pi_B.
\]

For simplicity we have ignored the isomorphism \((A \otimes B)^{**} \simeq A^{**} \otimes B^{**}\).
IV. Duals in monoidal categories

To check naturality, we perform the following calculation:

Conversely, we can use a pivotal structure to define a twist.
A symmetric monoidal category with duals has a canonical twist.

**Definition 49.** A *compact category* is a pivotal symmetric monoidal category with duals where the canonical twist is the identity \( \theta_A = \text{id}_A \).
A symmetric monoidal category with duals has a canonical twist.

**Definition 49.** A *compact category* is a pivotal symmetric monoidal category with duals where the canonical twist is the identity $\theta_A = \text{id}_A$.

Our example categories \textbf{FHilb}, \textbf{FVect} and \textbf{Rel} are all compact categories.
IV. Duals in monoidal categories

A symmetric monoidal category with duals has a canonical twist.

**Definition 49.** A compact category is a pivotal symmetric monoidal category with duals where the canonical twist is the identity \( \theta_A = \text{id}_A \).

Our example categories \( \text{FHilb} \), \( \text{FVect} \) and \( \text{Rel} \) are all compact categories.

Note that *in general*, other balancings may exist: that is, it is possible for a symmetric monoidal category with duals and a twist *not* to be a compact category.
IV. Duals in monoidal categories

Lemma 50. In a compact category, the following equations hold:
IV. Duals in monoidal categories

Lemma 50. In a compact category, the following equations hold:

\[ \begin{align*}
\mathsf{U} & = \mathsf{X} \\
\mathsf{O} & = \mathsf{O}
\end{align*} \]

Proof. Let's prove the second equation in the top row:

\[ \begin{align*}
\varepsilon_A & \ast \pi_A \\
\varepsilon_A & \ast \eta_A \\
\varepsilon_A & \ast \varepsilon_A
\end{align*} \]

The others can be proved in a similar way.
IV. Duals in monoidal categories

In a braided pivotal category, we must be careful with loops:
IV. Duals in monoidal categories

In a braided pivotal category, we must be careful with loops:

\[
\theta \neq \theta^{-1}
\]

In fact, a loop on a single strand is directly related to the twist. **Lemma 51.** In a braided pivotal category, the following hold:

\[
\theta = \theta^{-1} = \theta
\]
Proof. Let’s verify the expression for $\theta^{-1}$:

\[
\begin{align*}
\begin{array}{c}
\theta \\
\theta^{-1}
\end{array}
&= \\

\begin{array}{c}
\theta \\
\theta^{-1}
\end{array}
&= \\

\begin{array}{c}
\text{iso} \\
\text{iso}
\end{array}
&= \\

\begin{array}{c}
\text{iso} \\
\text{iso}
\end{array}
&= \\

\end{align*}
\]
Proof. Let’s verify the expression for $\theta^{-1}$:

The equation $\theta \circ \theta^{-1} = \text{id}$ can be checked in a similar way. Since inverses in a category are unique, this proves $\theta^{-1}$ is correct.
IV. Duals in monoidal categories

Proof. Let’s verify the expression for $\theta^{-1}$:

\[
\theta^{-1} \circ \theta = \text{id}
\]

The equation $\theta \circ \theta^{-1} = \text{id}$ can be checked in a similar way. Since inverses in a category are unique, this proves $\theta^{-1}$ is correct.

We demonstrate the graphical form of $\theta^*$ as follows:

\[
\theta = \text{id}
\]

The rest of the theorem can be proved similarly.
Thinking about ribbons inspires the following definition.

**Definition 52.** A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_{A^*}$. 
IV. Duals in monoidal categories

Thinking about ribbons inspires the following definition.

**Definition 52.** A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_{A^*}$.

This is equivalent to either of these graphical equations:

![Diagram](attachment:image.png)
IV. Duals in monoidal categories

Thinking about ribbons inspires the following definition.

**Definition 52.** A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that $(\theta_A)^* = \theta_A^*$. 

This is equivalent to either of these graphical equations:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {}; 
  \node (B) at (1,0) {}; 
  \node (C) at (1,1) {}; 
  \node (D) at (0,1) {}; 
  \draw [->] (A) to (B); 
  \draw [->] (B) to (C); 
  \draw [->] (C) to (D); 
  \draw [->] (D) to (A); 
\end{tikzpicture}
\end{array}
&= \\
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {}; 
  \node (B) at (1,0) {}; 
  \node (C) at (1,1) {}; 
  \node (D) at (0,1) {}; 
  \draw [->] (A) to (B); 
  \draw [->] (B) to (C); 
  \draw [->] (C) to (D); 
  \draw [->] (D) to (A); 
  \draw [->] (B) to (D); 
\end{tikzpicture}
\end{array}
= \\
\begin{array}{c}
\begin{tikzpicture}
  \node (A) at (0,0) {}; 
  \node (B) at (1,0) {}; 
  \node (C) at (1,1) {}; 
  \node (D) at (0,1) {}; 
  \draw [->] (A) to (B); 
  \draw [->] (B) to (C); 
  \draw [->] (C) to (D); 
  \draw [->] (D) to (A); 
  \draw [->] (B) to (D); 
\end{tikzpicture}
\end{array}
\end{align*}
\]

**Lemma 53.** A compact category is a ribbon category.
IV. Duals in monoidal categories

Thinking about ribbons inspires the following definition.

**Definition 52.** A *ribbon* or *tortile* category is a balanced monoidal category with duals, such that \((\theta_A)^* = \theta_A^*\).

This is equivalent to either of these graphical equations:

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{align*}
\]

**Lemma 53.** A compact category is a ribbon category.

**Lemma 54.** In a ribbon category, the following equations hold:

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array}
\end{align*}
\]
IV. Duals in monoidal categories

These are the equations we would expect to be satisfied by *ribbons*.

**Theorem 55.** A well-formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in three dimensions.
IV. Duals in monoidal categories

These are the equations we would expect to be satisfied by *ribbons*.

**Theorem 55.** A well-formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in three dimensions.

‘Framed isotopy’ is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires.
IV. Duals in monoidal categories

These are the equations we would expect to be satisfied by *ribbons*.

**Theorem 55.** A well-formed equation between morphisms in a ribbon category follows from the axioms if and only if it holds in the graphical language up to framed isotopy in three dimensions.

‘Framed isotopy’ is the name for the version of isotopy where the strands are thought of as ribbons, rather than just wires.

To get a feeling for framed isotopy, use ribbons to verify the following equations:
Part V

Duals in higher categories
V. Duals in higher categories

**Definition.** In a monoidal 2-category, an object $A$ has a *right dual* $B$ when it can be equipped with 1-morphisms called *folds*.
V. Duals in higher categories

**Definition.** In a monoidal 2-category, an object $A$ has a *right dual* $B$ when it can be equipped with 1-morphisms called *folds* $\varepsilon$ and 2-morphisms called *cusps* $\eta$.

![Diagram](image)
V. Duals in higher categories

The invertibility equations look like this:

It’s just like deforming a piece of fabric!
V. Duals in higher categories

To capture all the structure of oriented manifolds, we must require that our fold morphisms *themselves* have duals.
To capture all the structure of oriented manifolds, we must require that our fold morphisms *themselves* have duals.

To see what happens, let’s investigate this duality:
V. Duals in higher categories

It has a unit and counit, which we draw like this:
It has a unit and counit, which we draw like this:

The snake equations for the duality then look like this:

Again, this makes sense in terms of deformations of surfaces!
V. Duals in higher categories

There is only one set of equations left to completely specify the behaviour of oriented surfaces. They look like this:
V. Duals in higher categories

There is only one set of equations left to completely specify the behaviour of oriented surfaces. They look like this:

These are called the *cusp-flip equations*. 
V. Duals in higher categories

There is only one set of equations left to completely specify the behaviour of oriented surfaces. They look like this:

These are called the *cusp-flip equations*.

The *Cobordism Hypothesis* says that you can describe $n$-dimensional manifolds in a similar way.