Graphical Conjunctive Queries

A completeness theorem for Cartesian bicategories

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IMT School for Advanced Studies Lucca

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Contents

SCHOOL FOR ADVANCED STUDIES LUCCA

- A graphical way of reasoning about monoidal categories

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  - 2-dimensional diagrams manipulated according to algebraic rules – hot research topic

String diagrams

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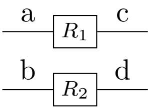

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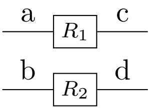
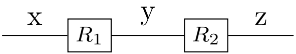


- Composition:

$$R_1 \,;\, R_2 = \{(x,z) \mid \exists y : (x,y) \in R_1, (y,z) \in R_2\}$$



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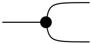
  - Copying and discarding 

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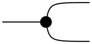
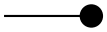
  - Copying and discarding  , 

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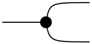
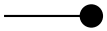
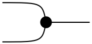
  - Copying and discarding  , 
  - Equality and "spawn" 

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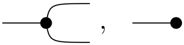
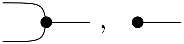
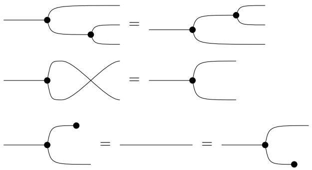
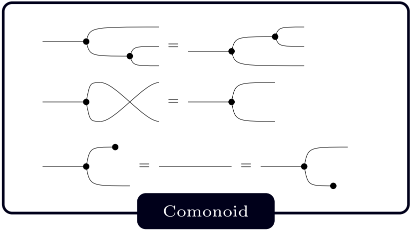
  - Copying and discarding 
  - Equality and "spawn" 

Observations



Comonoid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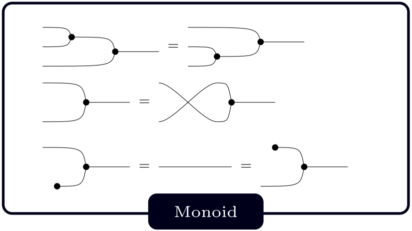
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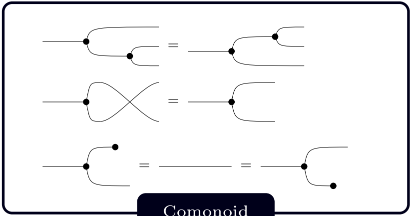

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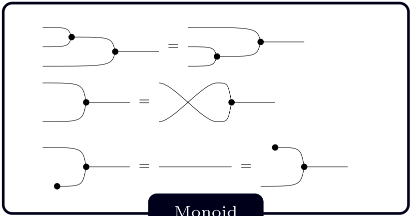

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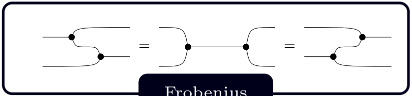

Frobenius

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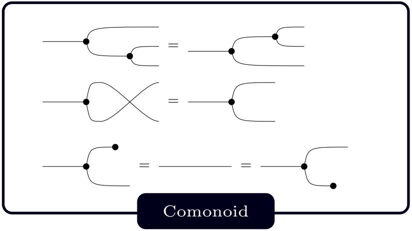

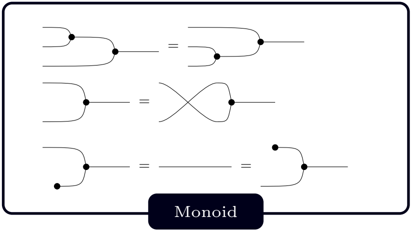

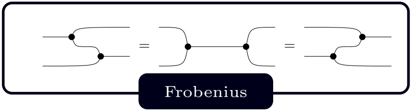

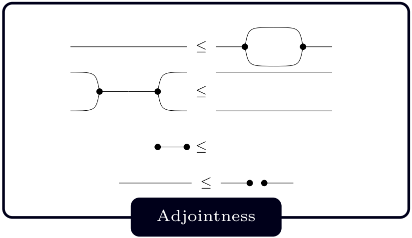

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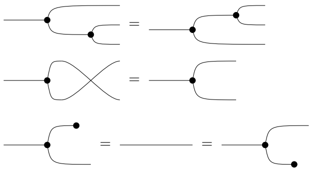

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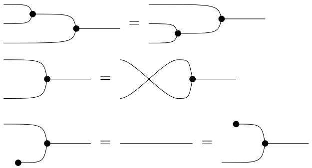

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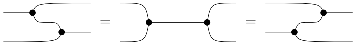

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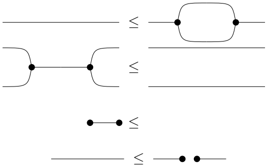
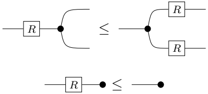


Adjointness



Lax Comonoid homomorphism

Definition (Carboni & Walters)

A Cartesian bicategory

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Idea: Do categorical logic with Cartesian bicategories.

Definition

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Not to be confused with "functional completeness"!

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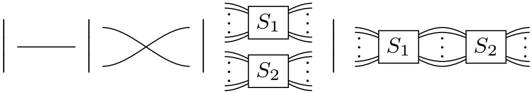

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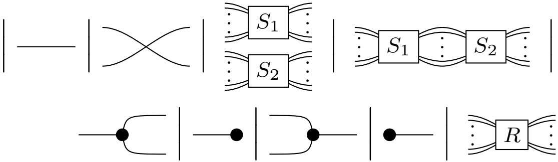

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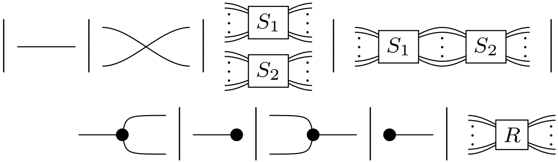

modulo the laws of Cartesian bicategories.

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Example

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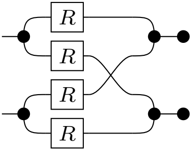

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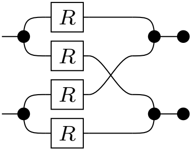

One-to-one correspondence between string diagrams and regular logic.

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- Query inclusion: $\phi \leq \psi$ iff $\mathcal{M}(\phi) \subseteq \mathcal{M}(\psi)$ in all models $\mathcal{M}$.

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$$\psi = \exists z_0, z_1 \colon R(x_0, z_0) \wedge R(x_1, z_0) \wedge R(x_0, z_1) \wedge R(x_1, z_1)$$

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A dictionary

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answer to query	semantics	$\mathcal{M}(S)$

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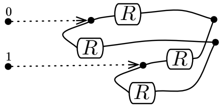

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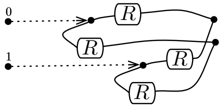

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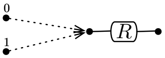

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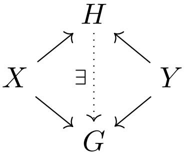

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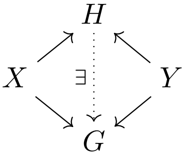

Dually define $\mathsf{Span}^{\sim}\mathcal{C}$ with all arrows reversed.

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Lemma

$$\mathbf{Rel} \cong \mathsf{Span}^\sim \mathbf{Set}$$

Theorem (Graphical theorem)

$$\mathbb{CB}_\Sigma \cong \mathsf{DiscCospan}^{\sim}\,\mathbf{Hyp}_\Sigma \subseteq \mathsf{Cospan}^{\sim}\,\mathbf{Hyp}_\Sigma$$

Lemma

$$\mathbf{Rel} \cong \mathsf{Span}^{\sim}\,\mathbf{Set}$$

Theorem (Completeness for $\mathbb{CB}_\Sigma$)

ϕ, ψ morphisms in $\mathbb{CB}_\Sigma$ such that $\mathcal{M}(\phi) \subseteq \mathcal{M}(\psi)$ for all morphisms $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$.
Then $\phi \leq \psi$.

Theorem (Graphical theorem)

$$\mathbb{CB}_\Sigma \cong \mathsf{DiscCospan}^\sim \mathbf{Hyp}_\Sigma \subseteq \mathsf{Cospan}^\sim \mathbf{Hyp}_\Sigma$$

Lemma

$$\mathbf{Rel} \cong \mathsf{Span}^\sim \mathbf{Set}$$

Theorem (Completeness for $\mathsf{Cospan}^\sim\mathcal{C}$)

ϕ, ψ morphisms in $\mathsf{Cospan}^\sim\mathcal{C}$ such that $\mathcal{M}(\phi) \leq \mathcal{M}(\psi)$ for all morphisms $\mathcal{M}\colon \mathsf{Cospan}^\sim\mathcal{C} \to \mathsf{Span}^\sim \mathbf{Set}$.
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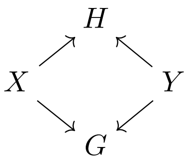

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$$
\begin{array}{ccccccccc}
 & H & & & & & \mathrm{Hom}(H,G) & & \\
 \nearrow & & \nwarrow & & & \swarrow & & \searrow & \\
X & & & Y & \overset{\mathrm{Hom}(\_,G)}{\rightsquigarrow} \quad \mathrm{Hom}(X,G) & & & & \mathrm{Hom}(Y,G) \\
 \searrow & & \swarrow & & & \nwarrow & & \nearrow & \\
 & G & & & & & \mathrm{Hom}(G,G) & &
\end{array}
$$

$\square$

Yoneda argument

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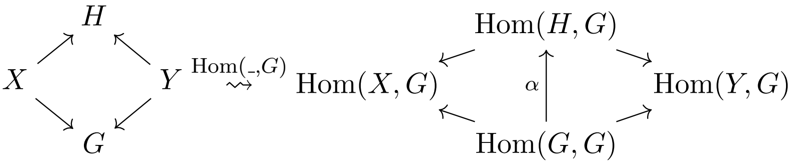

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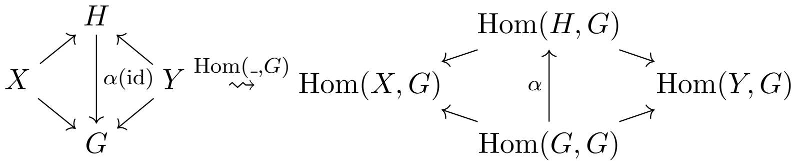

□

Corollary

The laws of Cartesian bicategories are sound and complete for query inclusion.

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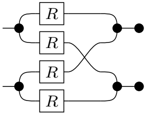
Example



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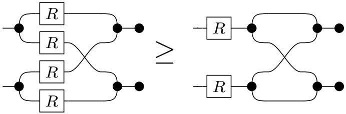

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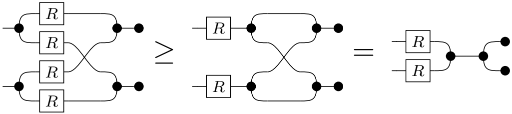

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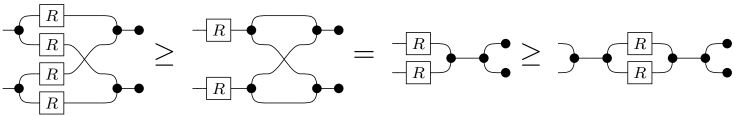

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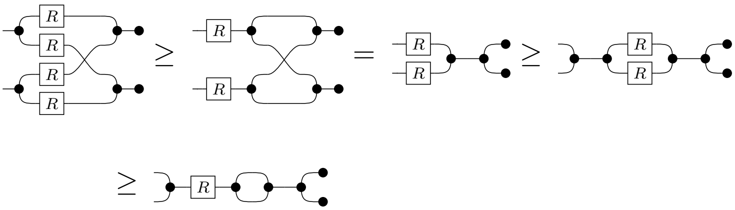

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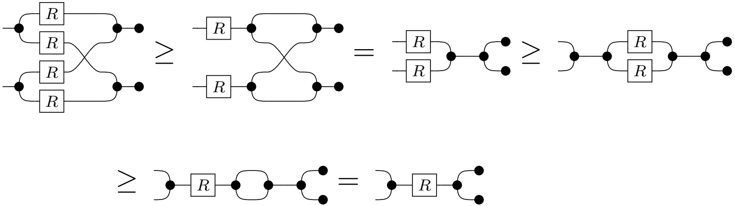
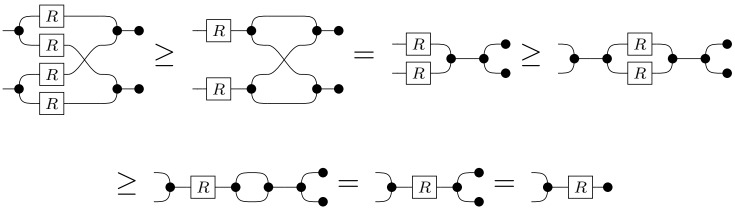


Corollary from Completeness

Corollary

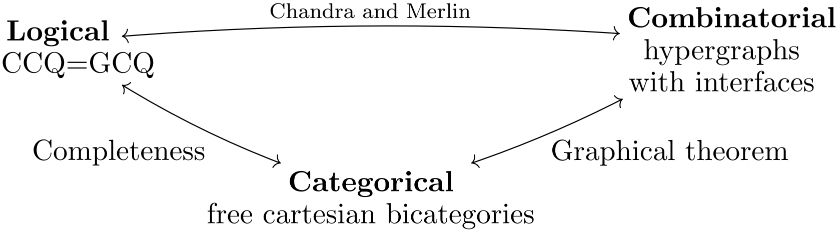
The laws of Cartesian bicategories are sound and complete for query inclusion. $\mathbb{CB}_\Sigma$ *is an algebra for conjunctive queries.*

Example



Contents

Summary


Chandra and Merlin
Logical
CCQ=GCQ
Combinatorial
hypergraphs
with interfaces
Completeness
Graphical theorem
Categorical
free cartesian bicategories

Theorem (hopefully coming soon)

Given morphisms x, y *in* $\mathcal{B}$ *such that* $\mathcal{M}(x) \subseteq \mathcal{M}(y)$ *for all* $\mathcal{M} \colon \mathcal{B} \to \mathbf{Rel}$. *Then*

$$x \leq y$$