Completeness for Cartesian bicategories

Relational algebra with string diagrams

Filippo Bonchi, *Jens Seeber*, Paweł Sobociński

IMT School for Advanced Studies Lucca

Glasgow - 18th December, 2018

SCHOOL FOR ADVANCED STUDIES LUCCA

Contents

- Idea: Use string diagrams as syntax for *relational* algebraic theories

- Idea: Use string diagrams as syntax for *relational* algebraic theories
- Develop a categorical logic for those theories

The category **Rel** of sets with relations as morphisms

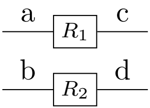
The category **Rel** of sets with relations as morphisms

- forms a symmetric monoidal category:

The category **Rel** of sets with relations as morphisms

- forms a symmetric monoidal category:

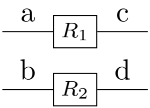
$$R_1 \otimes R_2 = \{((a,b),(c,d)) \mid (a,c) \in R_1, (b,d) \in R_2\}$$



The category **Rel** of sets with relations as morphisms

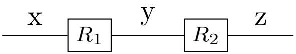
- forms a symmetric monoidal category:

$$R_1 \otimes R_2 = \{((a,b),(c,d)) \mid (a,c) \in R_1, (b,d) \in R_2\}$$



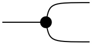
- Composition:

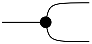
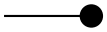
$$R_1 \,;\, R_2 = \{(x,z) \mid \exists y : (x,y) \in R_1, (y,z) \in R_2\}$$



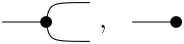
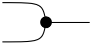
- Relations are ordered by inclusion

- Relations are ordered by inclusion
- Every object:

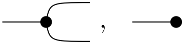
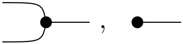
- Relations are ordered by inclusion
- Every object:
  - Copying and discarding 

- Relations are ordered by inclusion
- Every object:
  - Copying and discarding  , 

Relations with string diagrams

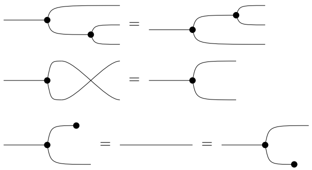
- Relations are ordered by inclusion
- Every object:
  - Copying and discarding 
  - Equality and "spawn" 

Relations with string diagrams

- Relations are ordered by inclusion
- Every object:
  - Copying and discarding 
  - Equality and "spawn" 

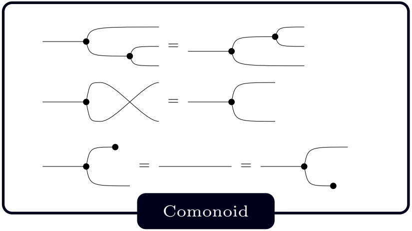
SCHOOL FOR ADVANCED STUDIES LUCCA

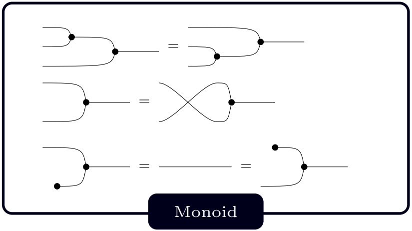
Observations



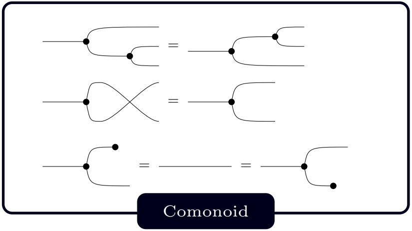
Comonoid

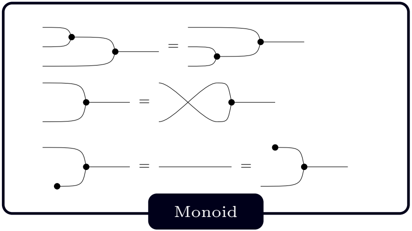
Observations

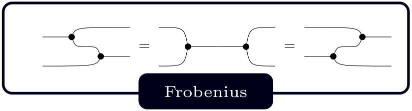




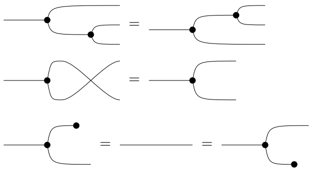
Observations



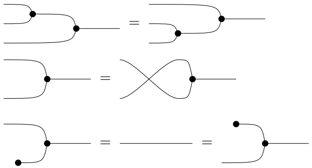




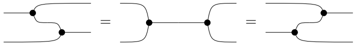
Observations



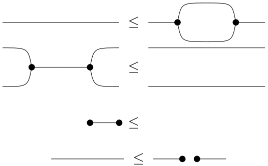
Comonoid



Monoid

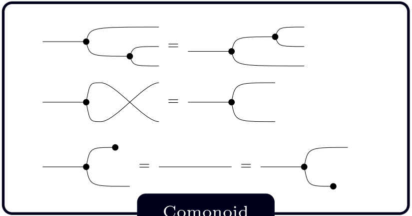


Frobenius

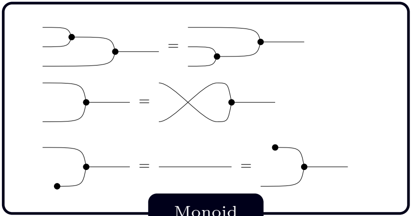


Adjointness

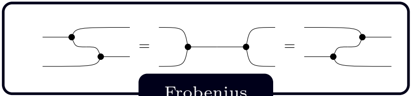
IMT SCHOOL FOR ADVANCED STUDIES LUCCA



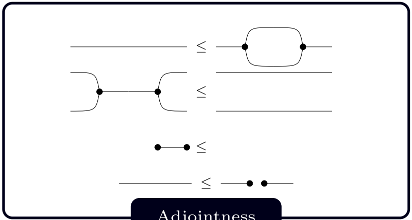
Comonoid



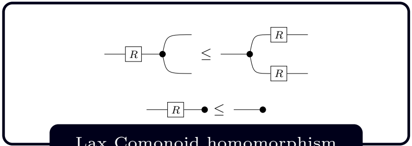
Monoid



Frobenius



Adjointness



Lax Comonoid homomorphism

Definition (Carboni & Walters)

A Cartesian bicategory

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence and the laws on the last slide.

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence and the laws on the last slide.

A morphism

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence and the laws on the last slide.

A morphism is a monoidal functor

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence and the laws on the last slide.

A morphism is a monoidal functor preserving the ordering, the comonoid and the monoid.

Definition (Carboni & Walters)

A Cartesian bicategory is a locally ordered symmetric monoidal category where every object is equipped with

- a comonoid
- a monoid

satisfying coherence and the laws on the last slide.

A morphism is a monoidal functor preserving the ordering, the comonoid and the monoid.

This captures the “relational algebraic” properties of **Rel**.

Contents

SCHOOL FOR ADVANCED STUDIES LUCCA

Definition

A Lawvere theory is a finite-product category with objects the natural numbers.

Definition

A Frobenius theory is a Cartesian bicategory with objects the natural numbers.

Definition

A Frobenius theory is a Cartesian bicategory with objects the natural numbers.

Definition

A model of a Lawvere theory T (in **Set**) is a morphism of finite-product categories

$$\mathcal{M}\colon T \to \mathbf{Set}$$

A morphism between models is a natural transformation.

SCHOOL FOR ADVANCED STUDIES LUCCA

Definition

A Frobenius theory is a Cartesian bicategory with objects the natural numbers.

Definition

A model of a Frobenius theory F (in **Rel**) is a morphism of Cartesian bicategories

$$\mathcal{M}\colon F \to \mathbf{Rel}$$

A morphism between models is a lax natural transformation.

Definition

A Frobenius theory is a Cartesian bicategory with objects the natural numbers.

Definition

A model of a Frobenius theory F (in $\mathbf{Rel}$) is a morphism of Cartesian bicategories

$$\mathcal{M} \colon F \to \mathbf{Rel}$$

A morphism between models is a lax natural transformation.

Theorem (Completeness for Lawvere theories)

If x, y are morphisms in T such that $\mathcal{M}(x) = \mathcal{M}(y)$ for all models $\mathcal{M} \colon T \to \mathbf{Set}$, then

$$x = y.$$

Definition

A Frobenius theory is a Cartesian bicategory with objects the natural numbers.

Definition

A model of a Frobenius theory F (in **Rel**) is a morphism of Cartesian bicategories

$$\mathcal{M}\colon F \to \mathbf{Rel}$$

A morphism between models is a lax natural transformation.

Theorem (Completeness for Frobenius theories)

If x, y are morphisms in F such that $\mathcal{M}(x) \leq \mathcal{M}(y)$ for all models $\mathcal{M}\colon F \to \mathbf{Rel}$, then

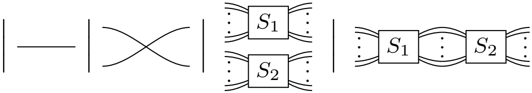
$$x \leq y.$$

Signature Σ

Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma \colon n \to m$.

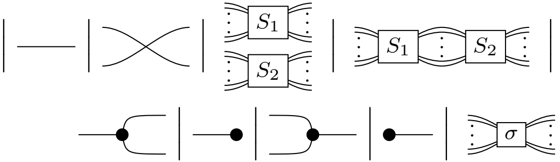
Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma\colon n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma \colon n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$ 

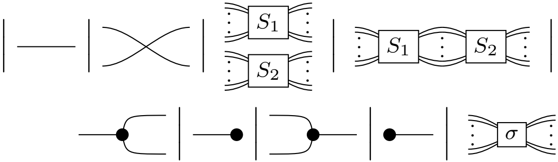
Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma \colon n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$



Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma : n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

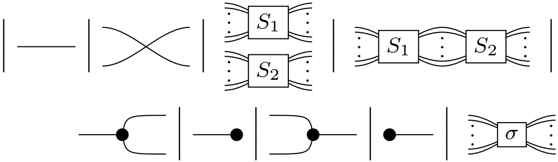
$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$



modulo the laws of Cartesian bicategories.

Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma\colon n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

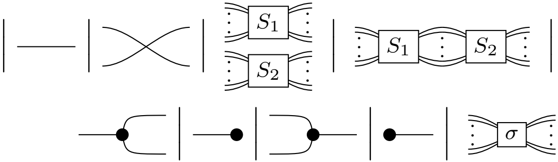
$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$



modulo the laws of Cartesian bicategories.
A model $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of

Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma : n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$

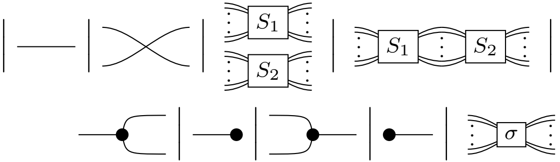


modulo the laws of Cartesian bicategories.
A model $\mathcal{M} : \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of

- a set $V = \mathcal{M}(1)$

Signature Σ, each $\sigma \in \Sigma$ equipped with arity and coarity, $\sigma\colon n \to m$.
Freely generated (syntactic) Cartesian bicategory $\mathbb{CB}_\Sigma$ has objects $\mathbb{N}$ and morphisms

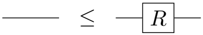
$\mathrm{Mor}(\mathbb{CB}_\Sigma) ::= \epsilon$



modulo the laws of Cartesian bicategories.
A model $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of

- a set $V = \mathcal{M}(1)$
- relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma\colon n \to m$

Example

- 

Example Frobenius theories

Example

- $\leq$ 

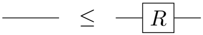
ensures that R is reflexive

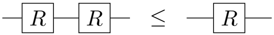
Example Frobenius theories

Example

- $—— \leq —\boxed{R}—$ ensures that R is reflexive
- $—\boxed{R}—\boxed{R}— \leq —\boxed{R}—$

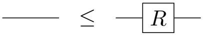
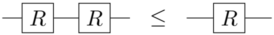
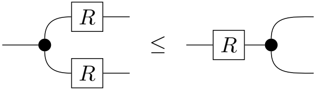
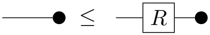
Example

- 

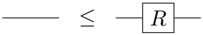
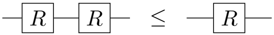
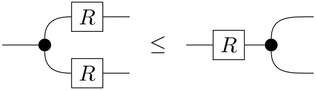
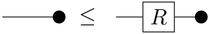
ensures that R is reflexive
- 

ensures that R is transitive

Example

-  ensures that R is reflexive
-  ensures that R is transitive
-  and 

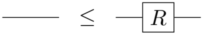
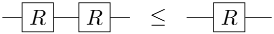
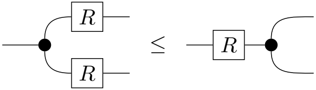
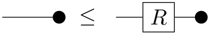
Example

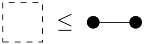
-  ensures that R is reflexive
-  ensures that R is transitive
-  and 

  ensure that R is a function.

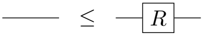
Example Frobenius theories

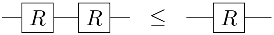
Example

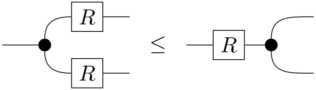
-  ensures that R is reflexive
-  ensures that R is transitive
-  and 

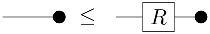
  ensure that R is a function.
- 

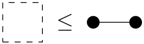
Example

- 

ensures that R is reflexive
- 

ensures that R is transitive
- 

and


ensure that R is a function.
- 

ensures that the underlying set is nonempty.

Definition

Fix a signature Σ and let E be a set of (well-typed) inequalities of morphisms in $\mathbb{CB}_\Sigma$.

Definition

Fix a signature Σ and let E be a set of (well-typed) inequalities of morphisms in $\mathbb{CB}_\Sigma$.
The Frobenius theory $\mathbb{CB}_{\Sigma/E}$ has the morphisms of $\mathbb{CB}_\Sigma$ taken modulo E.

Definition

Fix a signature Σ and let E be a set of (well-typed) inequalities of morphisms in $\mathbb{CB}_\Sigma$.
The Frobenius theory $\mathbb{CB}_{\Sigma/E}$ has the morphisms of $\mathbb{CB}_\Sigma$ taken modulo E.

Lemma

A model of $\mathbb{CB}_{\Sigma/E}$ is the same thing as a model of $\mathbb{CB}_\Sigma$ satisfying E.

Definition

Fix a signature Σ and let E be a set of (well-typed) inequalities of morphisms in $\mathbb{CB}_\Sigma$.
The Frobenius theory $\mathbb{CB}_{\Sigma/E}$ has the morphisms of $\mathbb{CB}_\Sigma$ taken modulo E.

Lemma

A model of $\mathbb{CB}_{\Sigma/E}$ is the same thing as a model of $\mathbb{CB}_\Sigma$ satisfying E.

Lemma

Every Frobenius theory is of the shape $\mathbb{CB}_{\Sigma/E}$ for some Σ, E.

Contents

SCHOOL FOR ADVANCED STUDIES LUCCA

- A model $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of
  - a set $V = \mathcal{M}(1)$
  - relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma \colon n \to m$

Σ-structures

- A model $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of
  - a set $V = \mathcal{M}(1)$
  - relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma\colon n \to m$
- In model theory, these are called Σ-structures.

SCHOOL FOR ADVANCED STUDIES LUCCA

- A model $\mathcal{M}\colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of
  - a set $V = \mathcal{M}(1)$
  - relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma\colon n \to m$
- In model theory, these are called Σ-structures.
- A morphism between Σ-structures is a function between the underlying sets respecting the relations.

- A model $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of
  - a set $V = \mathcal{M}(1)$
  - relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma \colon n \to m$
- In model theory, these are called Σ-structures.
- A morphism between Σ-structures is a function between the underlying sets respecting the relations.
- One can view a set as a Σ-structure – with empty relations

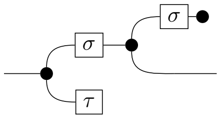
- A model $\mathcal{M} \colon \mathbb{CB}_\Sigma \to \mathbf{Rel}$ consists of
  - a set $V = \mathcal{M}(1)$
  - relations $\mathcal{M}(\sigma) \subseteq V^n \times V^m$ for $\sigma \in \Sigma$, $\sigma \colon n \to m$
- In model theory, these are called Σ-structures.
- A morphism between Σ-structures is a function between the underlying sets respecting the relations.
- One can view a set as a Σ-structure – with empty relations
- For S a Σ-structure, a morphism $n \to S$ is an n-tuple in S

Example

We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).

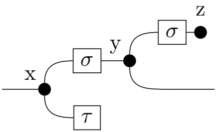
Example

We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).



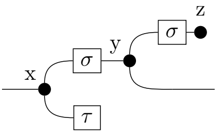
Example

We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).



Example

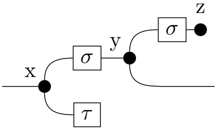
We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).



$$\mathcal{U}_R(1) = \{x, y, z\}$$

Example

We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).

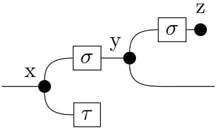


$$\mathcal{U}_R(1) = \{x, y, z\}$$

$$\mathcal{U}_R(\sigma) = \{(x, y), (y, z)\}, \ \ \mathcal{U}_R(\tau) = \{x\}$$

Example

We can translate a morphism $R\colon n \to m$ in $\mathbb{CB}_\Sigma$ to a finite model $\mathcal{U}_R$ with $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ (called *universal* model).



$$\mathcal{U}_R(1) = \{x, y, z\}$$

$$\mathcal{U}_R(\sigma) = \{(x, y), (y, z)\},\ \ \mathcal{U}_R(\tau) = \{x\}$$

$$\iota_R = x,\ \ \omega_R = y$$

SCHOOL FOR ADVANCED STUDIES LUCCA

Theorem (SYCO 1)

The assignment of $R\colon n \to m$ *to* $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ *is a bijection*

Theorem (SYCO 1)

The assignment of $R\colon n \to m$ *to* $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ *is a bijection between morphisms in* $\mathbb{CB}_\Sigma$ *and discrete cospans of finite* Σ*-structures.*

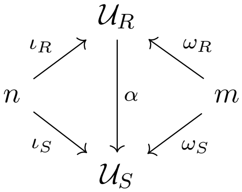
Connection with Completeness

Theorem (SYCO 1)

The assignment of $R\colon n \to m$ *to* $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ *is a bijection between morphisms in* $\mathbb{CB}_\Sigma$ *and discrete cospans of finite* Σ*-structures.*

$$S \leq R$$

if and only if there is $\alpha\colon \mathcal{U}_R \to \mathcal{U}_S$ *such that*



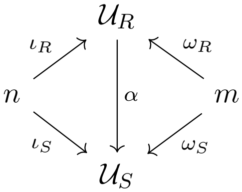
Connection with Completeness

Theorem (SYCO 1)

The assignment of $R\colon n \to m$ *to* $n \xrightarrow{\iota_R} \mathcal{U}_R \xleftarrow{\omega_R} m$ *is a bijection between morphisms in* $\mathbb{CB}_\Sigma$ *and discrete cospans of finite* Σ*-structures.*

$$S \leq R$$

if and only if there is $\alpha\colon \mathcal{U}_R \to \mathcal{U}_S$ *such that*



Connects semantics to syntax.

The $(\cdot)_E$ construction

Idea: Saturate a Σ-structure with respect to the axioms E.

Idea: Saturate a Σ-structure with respect to the axioms E.

Theorem

There is a functor $(\cdot)_E : \mathrm{Mod}_{\mathbb{CB}_\Sigma} \to \mathrm{Mod}_{\mathbb{CB}_\Sigma}$ *with a natural transformation* $\zeta_S \colon S \to S_E$ *with the following property:*

Idea: Saturate a Σ-structure with respect to the axioms E.

Theorem

There is a functor $(\cdot)_E : \mathrm{Mod}_{\mathbb{CB}_\Sigma} \to \mathrm{Mod}_{\mathbb{CB}_\Sigma}$ *with a natural transformation* $\zeta_S \colon S \to S_E$ *with the following property:*
If $A \leq B$ *is an axiom in* E*, and* $(x, y) \in S(A)$ *then* $(\zeta(x), \zeta(y)) \in S_E(B)$.

Idea: Saturate a Σ-structure with respect to the axioms E.

Theorem

There is a functor $(\cdot)_E : \mathrm{Mod}_{\mathbb{CB}_\Sigma} \to \mathrm{Mod}_{\mathbb{CB}_\Sigma}$ *with a natural transformation* $\zeta_S \colon S \to S_E$ *with the following property:*
If $A \leq B$ *is an axiom in* E*, and* $(x, y) \in S(A)$ *then* $(\zeta(x), \zeta(y)) \in S_E(B)$.

Definition

An algebra for the pointed endofunctor $(\cdot)_E$ is a Σ-structure S with a morphism $\alpha \colon S_E \to S$ such that $\alpha \circ \zeta_S = \mathrm{id}_S$

Idea: Saturate a Σ-structure with respect to the axioms E.

Theorem

There is a functor $(\cdot)_E : \mathrm{Mod}_{\mathbb{CB}_\Sigma} \to \mathrm{Mod}_{\mathbb{CB}_\Sigma}$ *with a natural transformation* $\zeta_S \colon S \to S_E$ *with the following property:*
If $A \leq B$ *is an axiom in* E*, and* $(x, y) \in S(A)$ *then* $(\zeta(x), \zeta(y)) \in S_E(B)$.

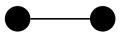
Definition

An algebra for the pointed endofunctor $(\cdot)_E$ is a Σ-structure S with a morphism $\alpha \colon S_E \to S$ such that $\alpha \circ \zeta_S = \mathrm{id}_S$

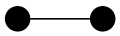
Lemma

$(\cdot)_E$*-algebras are models for* $\mathbb{CB}_{\Sigma/E}$.

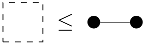
Example

Take $\Sigma = \varnothing$, $E = \Big\{$  $\leq$  $\Big\}$

Example

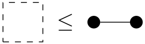
Take $\Sigma = \varnothing$, $E = \Big\{$  $\leq$  $\Big\}$, $\mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}}$ is the category of non-empty sets.

Example

Take $\Sigma = \varnothing$, $E = \Big\{$  $\Big\}$, $\mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}}$ is the category of non-empty sets.

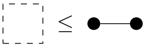
- $S_E = S + 1$

Example

Take $\Sigma = \varnothing$, $E = \{$  $\}$, $\text{Mod}_{\mathbb{CB}_{\Sigma/E}}$ is the category of non-empty sets.

- $S_E = S + 1$
- $(\cdot)_E$–algebras are *pointed* sets

Example

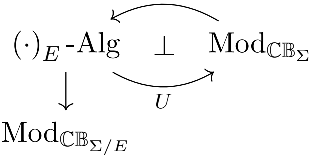
Take $\Sigma = \varnothing$, $E = \{$  $\}$, $\mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}}$ is the category of non-empty sets.

- $S_E = S + 1$
- $(\cdot)_E$–algebras are *pointed* sets

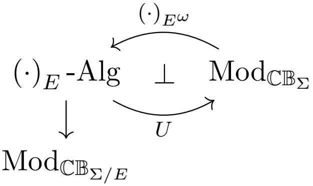
The category of $(\cdot)_E$–algebras is better behaved than $\mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}}$.

$$\begin{array}{ccc} (\cdot)_E\text{-Alg} & \xrightarrow[U]{} & \mathrm{Mod}_{\mathbb{CB}_\Sigma} \\ \downarrow & & \\ \mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}} & & \end{array}$$

The free algebra



The free algebra



The free algebra

$$\begin{array}{ccc} (\cdot)_E\text{-Alg} & \underset{U}{\overset{(\cdot)_{E^\omega}}{\leftrightarrows}}\ \perp & \mathrm{Mod}_{\mathbb{CB}_\Sigma} \\ \downarrow & & \\ \mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}} & & \end{array}$$

$$S \longrightarrow S_E \longrightarrow S_{E^2} \longrightarrow \cdots \longrightarrow S_{E^\omega}$$

The free algebra

$$
\begin{array}{ccc}
 & \xleftarrow{(\cdot)_{E^\omega}} & \\
(\cdot)_E\text{-Alg} & \perp & \mathrm{Mod}_{\mathbb{CB}_\Sigma} \\
\downarrow & \xrightarrow[U]{} & \\
\mathrm{Mod}_{\mathbb{CB}_{\Sigma/E}} & &
\end{array}
$$

$$
S \longrightarrow S_E \rightrightarrows S_{E^2} \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots \longrightarrow S_{E^\omega}
$$

Theorem

$$S \leq R \ \textit{in}\ \mathbb{CB}_\Sigma$$

if and only if there is $\alpha\colon \mathcal{U}_R \to \mathcal{U}_S$ *such that*

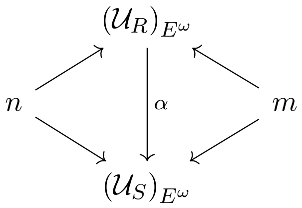
$$\begin{array}{ccccc} & & \mathcal{U}_R & & \\ & \nearrow^{\iota_R} & & \nwarrow^{\omega_R} & \\ n & & \downarrow \alpha & & m \\ & \searrow_{\iota_S} & & \swarrow_{\omega_S} & \\ & & \mathcal{U}_S & & \end{array}$$

From semantics to syntax

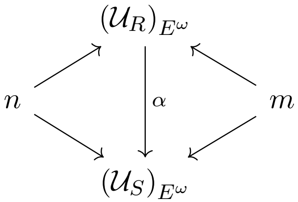
Theorem

$$S \leq R \textit{ in } \mathbb{CB}_{\Sigma/E}$$

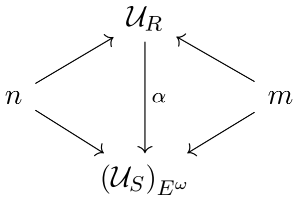
if and only if there is $\alpha\colon (\mathcal{U}_R)_{E^\omega} \to (\mathcal{U}_S)_{E^\omega}$ *such that*



Proof sketch.

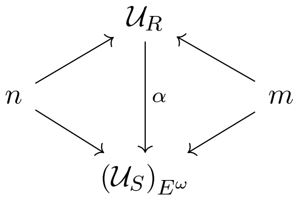


Proof sketch.



From semantics to syntax

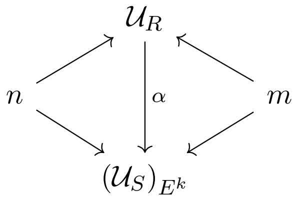
Proof sketch.



- $\mathcal{U}_R$ is compact

From semantics to syntax

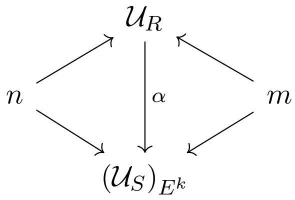
Proof sketch.



- $\mathcal{U}_R$ is compact

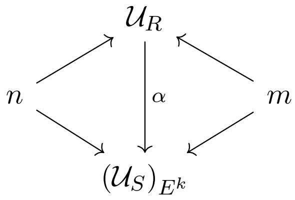
From semantics to syntax

Proof sketch.



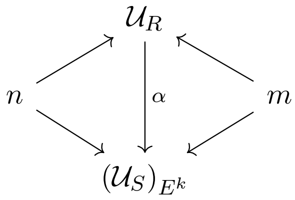
- $\mathcal{U}_R$ is compact
- $n \to (\mathcal{U}_S)_{E^i} \leftarrow m$ correspond to string diagrams S_i

Proof sketch.



- $\mathcal{U}_R$ is compact
- $n \to (\mathcal{U}_S)_{E^i} \leftarrow m$ correspond to string diagrams S_i
- $S_k \leq R$ in $\mathbb{CB}_\Sigma$, hence in $\mathbb{CB}_{\Sigma/E}$

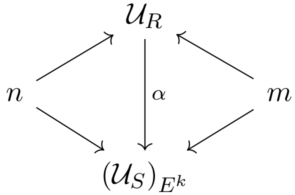
Proof sketch.



- $\mathcal{U}_R$ is compact
- $n \to (\mathcal{U}_S)_{E^i} \leftarrow m$ correspond to string diagrams S_i
- $S_k \leq R$ in $\mathbb{CB}_\Sigma$, hence in $\mathbb{CB}_{\Sigma/E}$
- S_{i+1} is obtained by blindly applying all axioms to S_i

From semantics to syntax

Proof sketch.



- $\mathcal{U}_R$ is compact
- $n \to (\mathcal{U}_S)_{E^i} \leftarrow m$ correspond to string diagrams S_i
- $S_k \leq R$ in $\mathbb{CB}_\Sigma$, hence in $\mathbb{CB}_{\Sigma/E}$
- S_{i+1} is obtained by blindly applying all axioms to S_i
- $S = S_0 \leq S_1 \leq S_2 \leq \cdots \leq S_k \leq R$ in $\mathbb{CB}_{\Sigma/E}$

Theorem

Frobenius theories are complete with respect to relational interpretations.