Superpositions and Categorical Quantum Reconstructions

Sean Tull

sean.tull@cs.ox.ac.uk
University of Oxford

SYCO 2
University of Strathclyde
December 17 2018
“Local and Global Phases in Categorical Quantum Theory”
The Plan

1. Motivation

2. Phased Biproducsts

3. Relating Local and Global Phases

4. Quantum Reconstructions
1. Motivation
Two Categories for Quantum Theory

Pure quantum theory is normally described via the category \( \text{Hilb} \) of Hilbert spaces and continuous linear maps \( f : \mathcal{H} \to \mathcal{K} \).

Question 1:
How is \( \text{Hilb} \) built from \( \text{Hilb}_P \)?
Pure quantum theory is normally described via the category $\text{Hilb}$ of Hilbert spaces and continuous linear maps $f: \mathcal{H} \to \mathcal{K}$.

But physically we only consider maps up to global phase:

$$f \sim g \iff f = z \cdot g \text{ for } z \in \mathbb{C}, \lvert z \rvert = 1.$$
Two Categories for Quantum Theory

Pure quantum theory is normally described via the category $\text{Hilb}$ of Hilbert spaces and continuous linear maps $f : \mathcal{H} \rightarrow \mathcal{K}$.

But physically we only consider maps up to global phase:

$$f \sim g \iff f = z \cdot g \text{ for } z \in \mathbb{C}, |z| = 1.$$  

Hence it is really given by the category

$$\text{Hilb}_P := \text{Hilb}/\sim$$

where morphisms are $\sim$-equivalence classes $[f] : \mathcal{H} \rightarrow \mathcal{K}$. 
Two Categories for Quantum Theory

Pure quantum theory is normally described via the category \textbf{Hilb} of Hilbert spaces and continuous linear maps $f : \mathcal{H} \rightarrow \mathcal{K}$.

But physically we only consider maps up to \textit{global phase}:

$$f \sim g \iff f = z \cdot g \text{ for } z \in \mathbb{C}, |z| = 1.$$

Hence it is really given by the category

$$\textbf{Hilb}_\mathcal{P} := \textbf{Hilb}/\sim$$

where morphisms are $\sim$-equivalence classes $[f] : \mathcal{H} \rightarrow \mathcal{K}$.

\textbf{Question 1}: How is \textbf{Hilb} built from \textbf{Hilb}_\mathcal{P}?
Idea: Superpositions

A defining quantum feature are superpositions, corresponding to an addition operation $f + g$ in $\text{Hilb}$. 
Idea: Superpositions

A defining quantum feature are superpositions, corresponding to an addition operation $f + g$ in $\text{Hilb}$.

This exists because $\text{Hilb}$ has biproducts:

$$
\mathcal{H} \xrightarrow{\kappa_1} \mathcal{H} \oplus \mathcal{K} \xleftarrow{\kappa_2} \mathcal{K}
$$
Idea: Superpositions

A defining quantum feature are superpositions, corresponding to an addition operation $f + g$ in $\text{Hilb}$.

This exists because $\text{Hilb}$ has biproducts:

\begin{align*}
\mathcal{H} & \xrightarrow{\kappa_1} \mathcal{H} \oplus \mathcal{K} & \xleftarrow{\kappa_2} \mathcal{K} \\
\xrightarrow{\pi_1} & \xleftarrow{\pi_2}
\end{align*}

This means that the $\kappa_i$ form a coproduct of $\mathcal{H}, \mathcal{K}$:

\begin{align*}
\mathcal{H} & \xrightarrow{\kappa_1} \mathcal{H} \oplus \mathcal{K} & \xleftarrow{\kappa_2} \mathcal{K} \\
\xrightarrow{\exists! h} & \xleftarrow{f, g}
\end{align*}

and the $\pi_i$ dually form a product, in a compatible way.
Idea: Superpositions

A defining quantum feature are superpositions, corresponding to an addition operation $f + g$ in $\textbf{Hilb}$.

This exists because $\textbf{Hilb}$ has biproducts:

\[
\begin{align*}
\mathcal{H} & \xrightarrow{\kappa_1} \mathcal{H} \oplus \mathcal{K} \xleftarrow{\kappa_2} \mathcal{K} \\
\pi_1 & \quad \pi_2
\end{align*}
\]

This means that the $\kappa_i$ form a coproduct of $\mathcal{H}, \mathcal{K}$:

\[
\begin{align*}
\mathcal{H} & \xrightarrow{\kappa_1} \mathcal{H} \oplus \mathcal{K} \xleftarrow{\kappa_2} \mathcal{K} \\
\mathcal{L} & \xrightarrow{f} \mathcal{H} \oplus \mathcal{K} \xrightarrow{g} \mathcal{K} \\
\end{align*}
\]

and the $\pi_i$ dually form a product, in a compatible way.

However: $\mathcal{H} \oplus \mathcal{K}$ is not a biproduct in $\textbf{Hilb}_P$. 
Question 2: How is $\mathcal{H} \oplus \mathcal{K}$ described in $\text{Hilb}_p$?
Question 2: How is $\mathcal{H} \oplus \mathcal{K}$ described in $\text{Hilb}_P$?

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{[\kappa_1]} & \mathcal{H} \oplus \mathcal{K} & \xleftarrow{[\kappa_2]} & \mathcal{K} \\
[f] & \downarrow{[h]} & & \downarrow{[g]} & \mathcal{L}
\end{array}
\]

Commutes when $h \circ \kappa_1 = z \cdot f$ and $h \circ \kappa_2 = w \cdot g$ for global phases $z, w$. 
**Question 2:** How is $\mathcal{H} \oplus \mathcal{K}$ described in $\text{Hilb}_P$?

\[
\begin{align*}
\mathcal{H} \xrightarrow{[\kappa_1]} & \mathcal{H} \oplus \mathcal{K} \xleftarrow{[\kappa_2]} \mathcal{K} \\
\downarrow{[f]} & \downarrow{[h]} & \downarrow{[g]} \\
\mathcal{L} & \\
\end{align*}
\]

Commutes when $h \circ \kappa_1 = z \cdot f$ and $h \circ \kappa_2 = w \cdot g$ for global phases $z, w$.

So $[h]$ exists but is now only unique up to a phase:

\[
\begin{align*}
\mathcal{H} \oplus \mathcal{K} \xrightarrow{[U]} & \mathcal{H} \oplus \mathcal{K} \quad \text{with} \quad U = \begin{pmatrix}
\text{id}_\mathcal{H} & 0 \\
0 & z \cdot \text{id}_\mathcal{K}
\end{pmatrix}
\end{align*}
\]
2. Phased Biproducts
Phased Coproducts

Definition

In any category, a phased coproduct of $A$, $B$ is an object $A \dot{+} B$ along with morphisms $\kappa_A$, $\kappa_B$ as below, called coprojections, such that:

1. For all $f$, $g$ as above there exists $h$ making the diagram commute;

2. For any such $h$, $h'$ we have $h' = h \circ U$ for some endomorphism $U$ of $A \dot{+} B$ which is a phase, meaning that $U \circ \kappa_A = \kappa_A U \circ \kappa_B = \kappa_B$. 


Phased Coproducts

Definition
In any category, a phased coproduct of $A, B$ is an object $A \dot{+} B$ along with morphisms $\kappa_A, \kappa_B$ as below, called coprojections, such that:

1. For all $f, g$ as above there exists $h$ making the diagram commute;
2. For any such $h$, $h' = h \circ U$ for some endomorphism $U$ of $A \dot{+} B$ which is a phase, meaning that $U \circ \kappa_A = \kappa_A U \circ \kappa_B = \kappa_B$.
Phased Coproducts

Definition

In any category, a phased coproduct of \( A, B \) is an object \( A \dot{+} B \) along with morphisms \( \kappa_A, \kappa_B \) as below, called coprojections, such that:

1. For all \( f, g \) as above there exists \( h \) making the diagram commute;

\[
\begin{array}{ccc}
A & \xrightarrow{\kappa_A} & A \dot{+} B \\
& \downarrow{f} & \downarrow{h} & \downarrow{g} \\
& C & \xleftarrow{\kappa_B} & B
\end{array}
\]
Phased Coproducts

Definition
In any category, a **phased coproduct** of $A, B$ is an object $A \hat{\vee} B$ along with morphisms $\kappa_A, \kappa_B$ as below, called **coprojections**, such that:

1. For all $f, g$ as above there exists $h$ making the diagram commute;
2. For any such $h, h'$ we have $h' = h \circ U$ for some endomorphism $U$ of $A \hat{\vee} B$ which is a **phase**, meaning that

$$U \circ \kappa_A = \kappa_A \quad U \circ \kappa_B = \kappa_B$$
Phased coproducts are surprisingly well-behaved.
Phased coproducts are surprisingly well-behaved.

Lemma

1. They are unique up to (non-unique) isomorphism.
2. Any phase is an isomorphism.
3. Associativity holds:

\[(A \dot{+} B) \dot{+} C \simeq A \dot{+} B \dot{+} C \simeq A \dot{+} (B \dot{+} C)\]

4. Having finite phased coproducts \(A_1 \dot{+} \cdots \dot{+} A_n\)
\[\iff\] having binary ones \(A \dot{+} B\) and an initial object 0.
Phased coproducts are surprisingly well-behaved.

**Lemma**

1. They are unique up to (non-unique) isomorphism.
2. Any phase is an isomorphism.
3. Associativity holds:
   \[(A + B) + C \simeq A + B + C \simeq A + (B + C)\]
4. Having finite phased coproducts \(A_1 + \cdots + A_n\)
   \(\iff\) having binary ones \(A + B\) and an initial object \(0\).

In a monoidal category, call them **distributive** when they interact well with \(\otimes\), via isomorphisms

\[A \otimes (B + C) \simeq (A \otimes B) + (A \otimes C)\]
Phased coproducts are surprisingly well-behaved.

Lemma

1. They are unique up to (non-unique) isomorphism.
2. Any phase is an isomorphism.
3. Associativity holds:

   $$(A + B) + C \simeq A + B + C \simeq A + (B + C)$$

4. Having finite phased coproducts $A_1 \cdots + A_n$

   $\iff$ having binary ones $A + B$ and an initial object $0$.

In a monoidal category, call them distributive when they interact well with $\otimes$, via isomorphisms

$$A \otimes (B + C) \simeq (A \otimes B) + (A \otimes C)$$

Can define phased products $(A \leftarrow A \times B \rightarrow B)$ dually, and even phased (co)limits more generally.
Phased Biproducts

**Definition**
In a category with zero morphisms, a phased biproduct of $A, B$ is an object $A \hat{\oplus} B$ which is both a phased coproduct and phased product:

$A \xleftarrow{\kappa_1} \xrightarrow{\pi_1} A \hat{\oplus} B \xleftarrow{\kappa_2} \xrightarrow{\pi_2} B$

In a dagger category, a phased dagger biproduct also has $\pi_i = \kappa_i^\dagger$. 

**Example**
$\text{Hilb} \mathcal{P}$ has phased dagger biproducts given by the direct sum $\mathcal{H} \oplus \mathcal{K}$. 

Phased Biproducts

Definition
In a category with zero morphisms, a phased biproduct of \( A, B \) is an object \( A \oplus B \) which is both a phased coproduct and phased product:

\[
A \xleftarrow{\kappa_1} \xrightarrow{\pi_1} A \oplus B \xleftarrow{\pi_2} B
\]

with the same phases for each, and satisfying

\[
\pi_i \circ \kappa_j = \begin{cases} 
\text{id} & i = j \\
0 & i \neq j 
\end{cases}
\]

In a dagger category, a phased dagger biproduct also has \( \pi_i = \kappa_i^\dagger \).
Phased Biproducts

Definition
In a category with zero morphisms, a phased biproduct of $A$, $B$ is an object $A \hat{\oplus} B$ which is both a phased coproduct and phased product:

$$\xymatrix{ A & A \hat{\oplus} B & B \ar[l]_{\kappa_1}^{\pi_1} \ar[r]_{\kappa_2}^{\pi_2} & \ar[l]_{\pi_1}^{\pi_2} }$$

with the same phases for each, and satisfying

$$\pi_i \circ \kappa_j = \begin{cases} \text{id} & i = j \\ 0 & i \neq j \end{cases}$$

In a dagger category, a phased dagger biproduct also has $\pi_i = \kappa_i^\dagger$. 
Phased Biproducts

Definition
In a category with zero morphisms, a phased biproduct of $A, B$ is an object $A \dot{\oplus} B$ which is both a phased coproduct and phased product:

$$A \xleftarrow{\pi_1} A \dot{\oplus} B \xrightarrow{\pi_2} B$$

with the same phases for each, and satisfying

$$\pi_i \circ \kappa_j = \begin{cases} 
\text{id} & i = j \\
0 & i \neq j 
\end{cases}$$

In a dagger category, a phased dagger biproduct also has $\pi_i = \kappa_i^\dagger$.

Example
$\text{Hilb}_P$ has phased dagger biproducts given by the direct sum $\mathcal{H} \oplus \mathcal{K}$ of Hilbert spaces.
3. Relating Local and Global Phases
From Global to Local Phases

In any monoidal category \((\mathcal{C}, \otimes)\), by a choice of global phases we mean a designated subgroup \(\mathbb{P}\) of its invertible central scalars.
From Global to Local Phases

In any monoidal category \((\mathcal{C}, \otimes)\), by a choice of global phases we mean a designated subgroup \(\mathbb{P}\) of its invertible central scalars. We then define

\[ f \sim g \iff f = p \cdot g \text{ for some } p \in \mathbb{P} \]

and write \(\mathcal{C}_\mathbb{P} := \mathcal{C}/\sim\).
From Global to Local Phases

In any monoidal category \((\mathbf{C}, \otimes)\), by a choice of \textit{global phases} we mean a designated subgroup \(\mathbb{P}\) of its invertible \textit{central} scalars. We then define

\[ f \sim g \iff f = p \cdot g \text{ for some } p \in \mathbb{P} \]

and write \(\mathbf{C}_{\mathbb{P}} := \mathbf{C}/\sim\).

\textbf{Lemma}

\textit{If \(\mathbf{C}\) has distributive (co,bi)products then \(\mathbf{C}_{\mathbb{P}}\) has distributive phased (co,bi)products.}
From Global to Local Phases

In any monoidal category \((\mathcal{C}, \otimes)\), by a choice of global phases we mean a designated subgroup \(\mathbb{P}\) of its invertible central scalars. We then define

\[ f \sim g \iff f = p \cdot g \text{ for some } p \in \mathbb{P} \]

and write \(\mathcal{C}_\mathbb{P} := \mathcal{C}/\sim\).

Lemma

If \(\mathcal{C}\) has distributive \((co,bi)\)products then \(\mathcal{C}_\mathbb{P}\) has distributive phased \((co,bi)\)products.

Examples

\(\text{Hilb}_\mathbb{P}\) has phased biproducts as we’ve seen, arising from \(\text{Hilb}\) via the global phases \(\mathbb{P} := \{z \in \mathbb{C} \mid |z| = 1\}\).

So does the quotient \(\text{Vec}_\mathbb{P}\) of \(\text{Vec} := k\)-vectors spaces and linear maps, via \(\mathbb{P} := \{\lambda \in k \mid \lambda \neq 0\}\).
From Local to Global Phases

Observation:

Linear maps \( f : \mathcal{H} \to \mathcal{K} \)

\[\iff\]  
Equivalence classes \( \left[ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \right] : \mathcal{H} \oplus \mathbb{C} \to \mathcal{K} \oplus \mathbb{C} \)
From Local to Global Phases

Observation:

\[ \text{Linear maps } f : \mathcal{H} \to \mathcal{K} \quad \iff \quad \text{Equivalence classes } \left[ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \right] : \mathcal{H} \oplus \mathbb{C} \to \mathcal{K} \oplus \mathbb{C} \]

Definition

Let \((\mathbf{D}, \otimes)\) have phased coproducts. We define a category \(\text{GP}(\mathbf{D})\) by:

- objects are phased coproducts of the form \(A = A \oplus I\) in \(\mathbf{D}\);
- morphisms are those \(f : A \to B\) in \(\mathbf{D}\) with:
From Local to Global Phases

Observation:

\[ \text{Linear maps } f : \mathcal{H} \to \mathcal{K} \iff \text{Equivalence classes } \left[ \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \right] : \mathcal{H} \oplus \mathbb{C} \to \mathcal{K} \oplus \mathbb{C} \]

Definition
Let \((\mathbf{D}, \otimes)\) have phased coproducts. We define a category \(\text{GP}(\mathbf{D})\) by:

- objects are phased coproducts of the form \(\mathbf{A} = \mathbf{A} \oplus \mathbf{I}\) in \(\mathbf{D}\);
- morphisms are those \(f : \mathbf{A} \to \mathbf{B}\) in \(\mathbf{D}\) with:

\[
\begin{align*}
\mathbf{A} & \xrightarrow{f} \mathbf{B} \\
\kappa_A & \uparrow \quad \kappa_B \\
\mathbf{A} \xrightarrow{g} \mathbf{B}
\end{align*}
\]
From Local to Global Phases

We have reached our first main result.

Theorem

Let $D$ be a monoidal category with finite distributive phased biproducts (resp. 'nice' phased coproducts). Then $\text{GP}(D)$ is a monoidal category with finite distributive biproducts (resp. coproducts) and a choice of global phases $P := \{u : I \to I | u$ is a phase on $I = I \hat{\otimes} I$ in $D\}$ such that $D \cong \text{GP}(D)_P$. 
From Local to Global Phases

We have reached our first main result.

**Theorem**

Let $D$ be a monoidal category with finite distributive phased biproducts (resp. ‘nice’ phased coproducts). Then $\text{GP}(D)$ is a monoidal category with finite distributive biproducts (resp. coproducts) and a choice of global phases

$$\mathbb{P} := \{ u: I \to I \mid u \text{ is a phase on } I = I + I \text{ in } D \}$$

such that

$$D \simeq \text{GP}(D)_\mathbb{P}$$
Summary

Biproducts and global phases

\[ \mathbf{C} \xrightarrow{\cong} \mathbf{GP(D)} \leftarrow \mathbf{D} \]

Phased Biproducets

\[ \mathbf{C}^p \]

Examples

\[ \text{Hilb} \cong \mathbf{GP(\text{Hilb})} \]

\[ \text{Vec} \cong \mathbf{GP(\text{Vec})} \]

Remark

Results generalise beyond monoidal setting, to categories:

1. \( \mathbf{C} \) with biproducts and trivial isomorphisms \( \cong \) on each object
2. \( \mathbf{D} \) with phased biproducts and a phase generator \( \mathbf{I} \).
Summary

Biproducts and global phases

\[ C \rightarrow \overset{\cong}{\rightarrow} \text{GP}(D) \leftarrow \text{GP}(C_P) \leftarrow D \]

Phased Biproducts

Examples

\[ \text{Hilb} \cong \text{GP}(\text{Hilb}_P) \]
\[ \text{Vec} \cong \text{GP}(\text{Vec}_P) \]
Summary

Biproducts and global phases

Phased Biproducts

\[ C \quad \text{C}_P \]

\[ \text{GP}(D) \quad D \]

Examples

\[ \text{Hilb} \simeq \text{GP}(\text{Hilb}_P) \]

\[ \text{Vec} \simeq \text{GP}(\text{Vec}_P) \]

Remark

Results generalise beyond monoidal setting, to categories:

- **C** with biproducts and trivial isomorphisms \( A \simeq A \) on each object \( A \)
- **D** with phased biproducts and a phase generator \( I \).
4. Quantum Reconstructions
Generalised Quantum Theories

If $D$ is a dagger compact category, then $\text{CPM}(D)$ has the same objects and morphisms $A \to B$ being those in $D$ of the form

![Diagram](attachment:image.png)
Generalised Quantum Theories

If \( D \) is a dagger compact category, then \( CPM(D) \) has the same objects and morphisms \( A \rightarrow B \) being those in \( D \) of the form

\[
\begin{array}{ccc}
B & C & B \\
\uparrow & & \uparrow \\
f & f & f \\
A & A & A
\end{array}
\]

\( \text{Mat}_S \): morphisms \( M: n \rightarrow m \) are \( m \times n \) matrices over \( S \), for any commutative involutive semi-ring \( (S, \dagger) \).
Generalised Quantum Theories

If $\mathcal{D}$ is a dagger compact category, then $\text{CPM}(\mathcal{D})$ has the same objects and morphisms $A \to B$ being those in $\mathcal{D}$ of the form

$$\begin{array}{c}
B \\
\downarrow \\
C \\
\downarrow \\
B
\end{array}
\quad
\begin{array}{c}
f \\
\downarrow \\
A
\end{array}
\quad
\begin{array}{c}
B \\
\downarrow \\
C \\
\downarrow \\
B
\end{array}
\quad
\begin{array}{c}
f \\
\downarrow \\
A
\end{array}$$

$\text{Mat}_S$: morphisms $M: n \to m$ are $m \times n$ matrices over $S$, for any commutative involutive semi-ring $(S, \dagger)$.

Definition

$\text{Quant}_S := \text{CPM}(\text{Mat}_S)$.
Generalised Quantum Theories

If $\mathbf{D}$ is a dagger compact category, then $\text{CPM}(\mathbf{D})$ has the same objects and morphisms $A \to B$ being those in $\mathbf{D}$ of the form

![Diagram](image)

$\text{Mat}_S$: morphisms $M: n \to m$ are $m \times n$ matrices over $S$, for any commutative involutive semi-ring $(S, \dagger)$.

**Definition**

$\text{Quant}_S := \text{CPM}(\text{Mat}_S)$.

**Examples**

$\text{Quant}_C$: fin. dim. Hilbert spaces and completely positive maps $f: B(\mathcal{H}) \to B(\mathcal{K})$.

$\text{Quant}_R$ is Quantum theory on real Hilbert spaces.
Lemma (Coecke)

A dagger compact $\mathbf{C}$ is of the form $\text{CPM}(\mathbf{D})$ precisely when it has an environment structure: a choice of
Lemma (Coecke)

* A dagger compact $\mathbf{C}$ is of the form $\mathrm{CPM}(\mathbf{D})$ precisely when it has an environment structure: a choice of
  - *discarding morphism* $\mathbb{T}_A$ on each object

Characterising Quantum Theories
Lemma (Coecke)

A dagger compact $\mathcal{C}$ is of the form $\text{CPM}(\mathcal{D})$ precisely when it has an environment structure: a choice of

- **discarding morphism** $\tilde{\tau}_A$ on each object
- **dagger compact subcategory** $\mathcal{C}_{\text{pure}}$ satisfying purification:

\[
\forall f \quad f = \begin{array}{c}
B \\
\downarrow f \\
A
\end{array} = \begin{array}{c}
B \\
\downarrow g \\
A
\end{array} \quad \text{for some } g \in \mathcal{D}_{\text{pure}}
\]

and some further axioms.
Characterising Quantum Theories

Lemma (Coecke)

A dagger compact $\mathbf{C}$ is of the form $\text{CPM}(\mathbf{D})$ precisely when it has an environment structure: a choice of

- discarding morphism $\top_A$ on each object
- dagger compact subcategory $\mathbf{C}_{\text{pure}}$ satisfying purification:

$$\forall f \quad f_A = g_A$$

for some $g \in \mathbf{D}_{\text{pure}}$

and some further axioms.

We will say that $\mathbf{C}_{\text{pure}}$ has the superposition properties when it has finite phased dagger biproducts satisfying some mild conditions.
A Recipe for Quantum Reconstructions
A Recipe for Quantum Reconstructions

**Theorem**

Let $(\mathcal{C}, \mathcal{C}_{\text{pure}}, \hat{\cdot})$ be an environment structure for which $\mathcal{C}_{\text{pure}}$ has the superposition properties. Then there is an embedding

$$\text{Quant}_S \hookrightarrow \mathcal{C}$$

preserving $\dagger, \otimes, \hat{\cdot}$, for some involutive semi-ring $S$ with $\mathcal{C}_{\text{pure}}(I, I) \simeq S^{\text{pos}}$. 
A Recipe for Quantum Reconstructions

Theorem

Let \((\mathbf{C}, \mathbf{C}_{\text{pure}}, \dagger)\) be an environment structure for which \(\mathbf{C}_{\text{pure}}\) has the superposition properties. Then there is an embedding

\[
\text{Quant}_S \hookrightarrow \mathbf{C}
\]

preserving \(\dagger, \otimes, \dagger\), for some involutive semi-ring \(S\) with \(\mathbf{C}_{\text{pure}}(I, I) \sim S^{\text{pos}}\).

Proof.

\(\text{GP}(\mathbf{C}_{\text{pure}})\) has biproducts, so contains \(\text{Mat}_S\) for its scalars \(S\). Then

\[
\text{Quant}_S \hookrightarrow \text{CPM}(\text{GP}(\mathbf{C}_{\text{pure}})) \sim_* \text{CPM}(\mathbf{C}_{\text{pure}}) \sim \mathbf{C}
\]

where \(\sim_*\) follows from our assumptions on \(\mathbf{C}_{\text{pure}}\). 

\[\square\]
Outlook

Phased co/biproducts:

▶ Describe superpositions in Hilb $P$;
▶ Allow passing to a 'nicer' category GP$(C)$, such as Hilb $C$;
▶ Provide a 'recipe' for reconstructing quantum-like theories.

However they are new and yet to be fully explored:
▶ Further (non-monoidal) examples throughout mathematics?
▶ Relation to other notions of weak (2-)limit?
▶ Generalisations of the GP construction?

Thanks for listening!
Outlook

Phased co/biproducts:

- Describe superpositions in $\text{Hilb}_P$;
Outlook

Phased co/biproducts:
- Describe superpositions in $\text{Hilb}_P$;
- Allow passing to a ‘nicer’ category $\text{GP}(\mathcal{C})$, such as $\text{Hilb}$;
Outlook

Phased co/biproducts:

► Describe superpositions in $\text{Hilb}_P$;
► Allow passing to a ‘nicer’ category $\text{GP}(\mathbf{C})$, such as $\text{Hilb}$;
► Provie a ‘recipe’ for reconstructing quantum-like theories.

However they are new and yet to be fully explored:
Phased co/biproducts:
- Describe superpositions in $\text{Hilb}_P$;
- Allow passing to a ‘nicer’ category $\text{GP}(C)$, such as $\text{Hilb}$;
- Provide a ‘recipe’ for reconstructing quantum-like theories.

However they are new and yet to be fully explored:
- Further (non-monoidal) examples throughout mathematics?
Outlook

Phased co/biproducts:

- Describe superpositions in $\text{Hilb}_P$;
- Allow passing to a ‘nicer’ category $\text{GP}(\mathbf{C})$, such as $\text{Hilb}$;
- Provie a ‘recipe’ for reconstructing quantum-like theories.

However they are new and yet to be fully explored:

- Further (non-monoidal) examples throughout mathematics?
- Relation to other notions of weak (2-)limit?
Outlook

Phased co/biproducts:
- Describe superpositions in $\text{Hilb}_P$;
- Allow passing to a ‘nicer’ category $\text{GP}(\mathcal{C})$, such as $\text{Hilb}$;
- Provide a ‘recipe’ for reconstructing quantum-like theories.

However they are new and yet to be fully explored:
- Further (non-monoidal) examples throughout mathematics?
- Relation to other notions of weak (2-)limit?
- Generalisations of the GP construction?
Outlook

Phased co/biproducts:

- Describe superpositions in $\text{Hilb}_P$;
- Allow passing to a ‘nicer’ category $\text{GP}(\mathbf{C})$, such as $\text{Hilb}$;
- Provide a ‘recipe’ for reconstructing quantum-like theories.

However they are new and yet to be fully explored:

- Further (non-monoidal) examples throughout mathematics?
- Relation to other notions of weak (2-)limit?
- Generalisations of the GP construction?

Thanks for listening!
References