

Process Matrices @ SYCO 2

Sander Uijlen

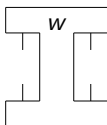


DEPARTMENT OF
**COMPUTER
SCIENCE**

18 December - Glasgow



Say something about process matrices
on general grounds
using categorical semantics
for higher order processes.



Process theories

Symmetric monoidal categories +
interpretation as systems and processes.

$$1_A := \begin{array}{c} | \\ \hline A \end{array} \quad f : A \rightarrow B := \begin{array}{c} | \\ \hline \boxed{f} \\ \hline A \end{array} \begin{array}{c} B \\ | \end{array}$$

$$g \circ f := \begin{array}{c} | \\ \hline \boxed{g} \\ \hline \boxed{f} \\ \hline | \end{array} \quad f \otimes g := \begin{array}{c} | \\ \hline \boxed{f} \\ \hline | \end{array} \begin{array}{c} | \\ \hline \boxed{g} \\ \hline | \end{array}$$

$$1_I := \boxed{} \quad \sigma_{A,B} := \begin{array}{c} | \quad | \\ \hline \\ \hline | \quad | \end{array} \begin{array}{c} B \quad A \\ | \quad | \\ \hline A \quad B \end{array}$$

States and effects

$$\begin{array}{c} | \\ \nabla \\ \rho \end{array} : I \rightarrow A \quad \textit{state}$$

$$\begin{array}{c} \triangle \\ \pi \\ | \end{array} : A \rightarrow I \quad \textit{effect}$$

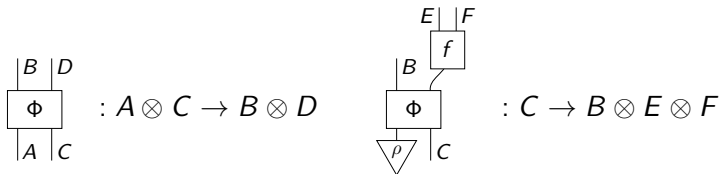
$$\lambda : I \rightarrow I \quad \textit{scalars}$$

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Only connectivity matters!

discarding

What else do we need to talk about causal orders?

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Consider a special family of *discarding* effects:

$$\overline{\top}_A \quad \overline{\top}_{A \otimes B} := \overline{\top}_A \overline{\top}_B \quad \overline{\top}_I := 1$$

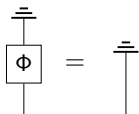
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This enables us to say when a process is *causal*:


$$\overline{\top} \text{ --- } \boxed{\Phi} \text{ --- } \overline{\top} = \overline{\top}$$

“If the outputs of a process are ignored, it doesn’t matter which process happened.”

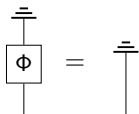
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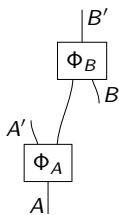
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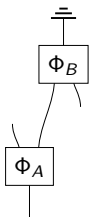
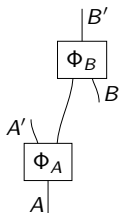
“If the outputs of a process are ignored, it doesn’t matter which process happened.”

Consequence: A causal process only affects other processes which consume its outputs (i.e., those that lie in its *causal future*).

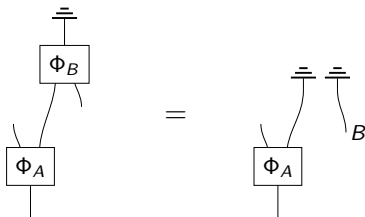
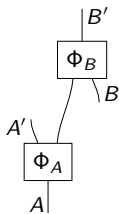
No signalling from the future



No signalling from the future

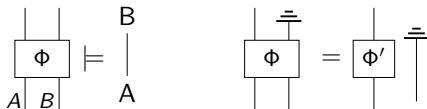


No signalling from the future



Causal orders

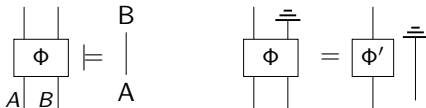
One-way signalling:



The output of A does not depend on B .

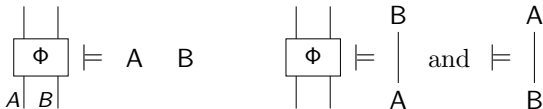
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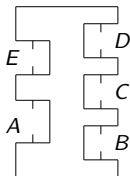
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No-signalling



Dual processes

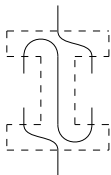
Causal order of dual process, e.g.,



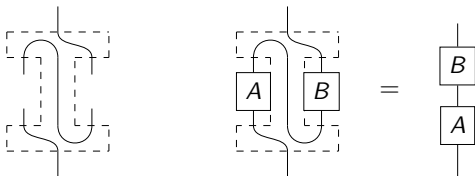
"What can they take in?"

Dual processes can witness the causal order.

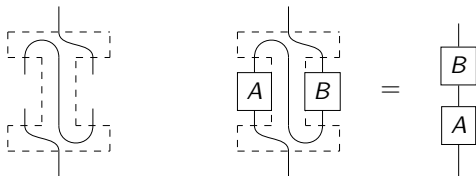
Higher Order Processes



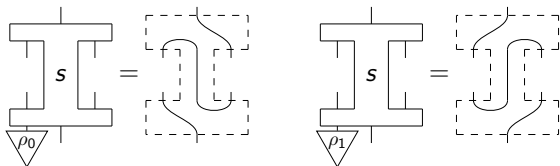
Higher Order Processes



Higher Order Processes



Quantum switch:



Allows for 'coherent superpositions' of causal orders.

Compact closure

An easy way to get higher-order processes!

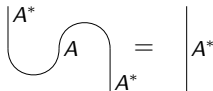
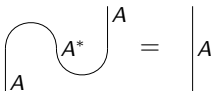
Compact closure

An easy way to get higher-order processes!

A way to 'bend wires'
(Choi-Jamiołkowski - Objects have duals A^*)

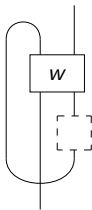


Satisfying



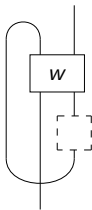
Problem

Take (causal) processes as inputs

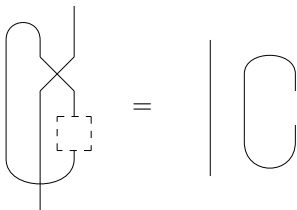


Problem

Take (causal) processes as inputs



...but this does not preserve causality:



Solution

Precausal category \mathcal{C} \mapsto

$\text{Caus}[\mathcal{C}]$

compact closed category
+ 4 $\overline{\top}$ axioms

**-autonomous category*
capturing 'logic of causality'

Solution

Precausal category \mathcal{C} \mapsto $\text{Caus}[\mathcal{C}]$

compact closed category
+ 4 $\overset{=}{\perp}$ axioms

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capturing 'logic of causality'

$\text{Mat}(\mathbb{R}_+)$
CPM

\mapsto
 \mapsto

HO stochastic maps
HO quantum channels

Input: Precausal Category

Compact closed category with $\overset{\equiv}{\vdash}$ + rules

- *Second-order causal processes factorise:*

$$\left(\forall \Phi \text{ causal} . \left(\text{Diagram} \right) = \overset{\equiv}{\vdash} \right) \Rightarrow \left(\exists \Phi_1, \Phi_2 \text{ causal} . \left(\text{Diagram} \right) = \text{Diagram} \right)$$

The diagram illustrates the factorization of a second-order causal process. On the left, a box labeled w contains a box labeled Φ . Above this box is a double line representing the multiplication of the box by the identity, with the text $\forall \Phi \text{ causal} .$ above it. This is shown to be equal to the double line representing the identity $\overset{\equiv}{\vdash}$. An arrow points to the right, where a similar box labeled w is shown to be equal to a box labeled Φ_1 connected to a box labeled Φ_2 . This is shown under an existential quantifier $\exists \Phi_1, \Phi_2 \text{ causal} .$.

Equivalent to

- Causal *one-way signalling* processes factorise:

$$\left(\begin{array}{c} \exists \Phi' \text{ causal .} \\ \text{Diagram: } \Phi = \Phi' \end{array} \right) \Rightarrow \left(\begin{array}{c} \exists \Phi_1, \Phi_2 \text{ causal .} \\ \text{Diagram: } \Phi = \text{Diagram with } \Phi_1 \text{ and } \Phi_2 \end{array} \right)$$

The diagram on the left shows a box labeled Φ with two input wires on the left and two output wires on the right. A double horizontal line with a bar is drawn above the top output wire, and another double horizontal line with a bar is drawn below the bottom output wire. This is equated to a box labeled Φ' with the same wire configuration.

The diagram on the right shows a box labeled Φ with the same wire configuration. To its right, there are two boxes: Φ_1 is positioned below the bottom output wire, and Φ_2 is positioned above the top output wire. This is equated to the original Φ box.

(Semicausal operations are semilocalizable)

- For all $w : A \otimes B^*$:

$$\left(\begin{array}{c} \forall \Phi \text{ causal .} \\ \text{Diagram: } w \text{ with } \Phi = 1 \end{array} \right) \Rightarrow \left(\begin{array}{c} \exists \rho \text{ causal .} \\ \text{Diagram: } w \text{ with } \rho \end{array} \right)$$

The diagram on the left shows a large box labeled w with four input wires on the left and four output wires on the right. Inside this box, there is a smaller box labeled Φ with two input wires on the left and two output wires on the right. The entire diagram is equated to the number 1.

The diagram on the right shows a large box labeled w with the same wire configuration. Inside, there is a smaller box labeled ρ with two input wires on the left and two output wires on the right. The entire diagram is equated to a double horizontal line with a bar above it.

Output: Category of Higher Order Processes - $\text{Caus}[\mathcal{C}]$

Build a new process theory describing 'causal' states:

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New types:

$$\mathbf{A} := (A, c_A) \text{ where } c_A \subseteq \mathcal{C}(I, A)$$

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With normalization and $c_{\mathbf{A}} = c_{\mathbf{A}}^{**}$

$$c^* := \left\{ \pi : \mathbf{A}^* \mid \forall \rho \in c . \begin{array}{c} \triangle \pi \\ \hline \rho \\ \nabla \end{array} = 1 \right\}$$

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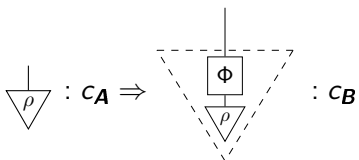
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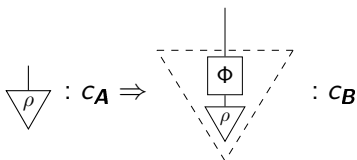
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Types: causal states, causal processes, no-signalling processes...

Processes preserve the *generalized* causal states:



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- Causal states to causal states,
- Causal processes to a number (dual),
- No-signalling processes to causal processes,
- ...

Caus[\mathcal{C}] is ISOMix *-autonomous

We have a tensor, unit and duals

$$\mathbf{A} \otimes \mathbf{B} := (A \otimes B, (c_A \otimes c_B)^{**})$$

$$\mathbf{I} := (I, \{1_I\}) \cong \mathbf{I}^*$$

$$\mathbf{A}^* := (A^*, c_A^*)$$

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...But no compact closure!

$$A \wp B := (A^* \otimes B^*)^* \neq A \otimes B$$

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Define $A \multimap B := A^* \wp B$ giving internal hom

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, B \multimap C)$$

In particular $\text{Hom}(A, B) \cong \text{Hom}(I, A \multimap B)$

Examples

First order

$$(A, \{\bar{\tau}_A\}^*)$$



Examples

First order $(A, \{\ddot{\tau}_A\}^*)$



First order dual $(A^*, \{\ddot{\tau}_A\})$

$\ddot{\tau}_A$

Examples

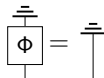
First order $(A, \{\ddot{\top}_A\}^*)$



First order dual $(A^*, \{\ddot{\top}_A\})$



Causal process $A \multimap B$



Examples

First order

$$(A, \{\bar{\tau}_A\}^*)$$



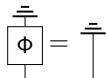
First order dual

$$(A^*, \{\bar{\tau}_A\})$$



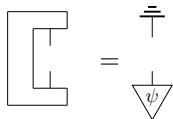
Causal process

$$A \multimap B$$



Dual of map

$$[A \multimap B]^*$$



Types Capture Causal Orders

Bipartite process

$$(\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$$



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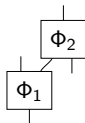
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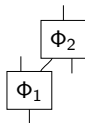
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No signalling

$$(A \multimap A') \otimes (B \multimap B') \quad \text{both factorizations}$$

Types Capture Causal Orders

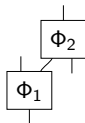
Bipartite process

$$(\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$$



One-way signalling

$$\mathbf{A} \multimap ((\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}')$$



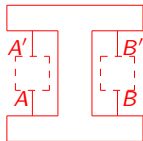
No signalling

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$$

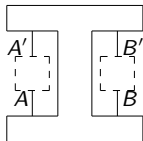
both factorizations

Dual-NS (W-matrix)

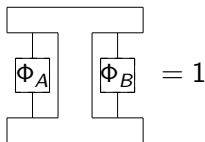
$$[(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')]^*$$



Process Matrices

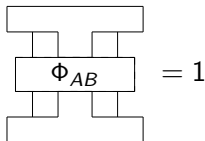


Satisfying for all causal Φ_A, Φ_B :



Process Matrices

Equivalent to normalization on all no-signalling processes:
for every no-signalling map $\Phi_{AB} : (\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$ we
have

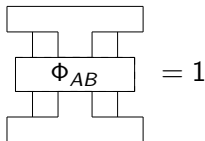


The diagram shows a central box labeled Φ_{AB} . From the top of this box, two vertical lines extend upwards to a horizontal bar. From the bottom of the box, two vertical lines extend downwards to another horizontal bar. To the right of this diagram is the equals sign followed by the number 1.

$$\Phi_{AB} = 1$$

Process Matrices

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Definition

A process matrix is a process in the dual of no-signalling processes, i.e., a process of type $[(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')]^*$.



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- Interesting informational properties

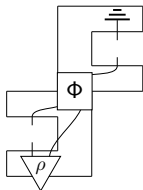
Indefinite causal orders

- Most general way to obtain probabilities from no-signalling processes.
- Interesting informational properties
- Can break causal bounds!
Quantum correlations with no causal order
Ognyan Oreshkov, Fabio Costa, Časlav Brukner -
arXiv:1105.4464v3

Duals of one-way signalling processes

Process matrices compatible with a specific causal order

$$(\mathbf{A} \preceq \mathbf{B})^* := [\mathbf{A} \text{---} \circ ((\mathbf{A}' \text{---} \circ \mathbf{B}) \text{---} \circ \mathbf{B}')]^*$$



Make a type which includes both duals $(\mathbf{A} \preceq \mathbf{B})^*$ and $(\mathbf{B} \preceq \mathbf{A})^*$

Proposition

The intersection of one-way signalling maps with $A \preceq B$ and one-way signalling maps $B \preceq A$ are the no-signalling maps.

$$\mathbf{A \preceq B \cap B \preceq A = NS}$$

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Theorem

$$[(\mathbf{A} \preceq \mathbf{B})^* \cup (\mathbf{B} \preceq \mathbf{A})^*]^{**} = \mathbf{NS}^*$$

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Theorem

$$[(\mathbf{A} \preceq \mathbf{B})^* \cup (\mathbf{B} \preceq \mathbf{A})^*]^{**} = \mathbf{NS}^*$$

Smallest type that contains both duals are all process matrices.

What does it mean..?

- In QM and probability theory, the double dual is the positive affine closure.
- Every process matrix is an affine closure of duals of one-way signalling processes (combs).

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- There are processes which are not *convex* combinations. (Switch, OCB, also in probability theory)

$$\frac{\textit{definite}}{\textit{indefinite}} \simeq \frac{\textit{seperable}}{\textit{entangled}}$$

Transformations of W-matrices

Type of such a transformation:

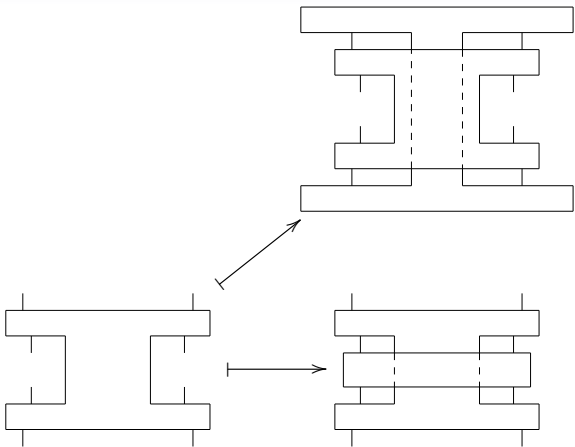
$$[(\mathbf{A} \circ \mathbf{A}') \otimes (\mathbf{B} \circ \mathbf{B}')]^* \circ [(\mathbf{C} \circ \mathbf{C}') \otimes (\mathbf{D} \circ \mathbf{D}')]^*$$

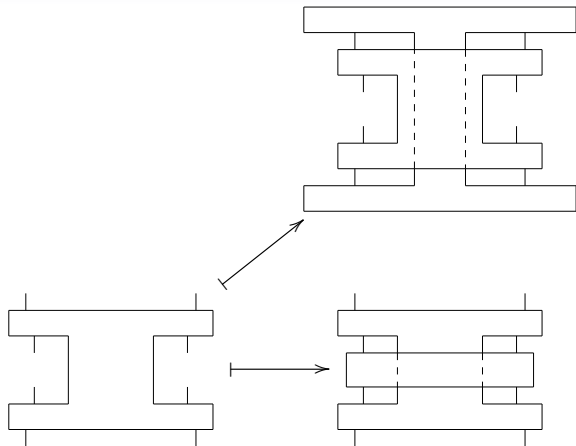
Transformations of W-matrices

Type of such a transformation:

$$\begin{aligned} & [(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')]^* \multimap [(\mathbf{C} \multimap \mathbf{C}') \otimes (\mathbf{D} \multimap \mathbf{D}')]^* \\ & \quad \cong \\ & [(\mathbf{C} \multimap \mathbf{C}') \otimes (\mathbf{D} \multimap \mathbf{D}')] \multimap [(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')] \end{aligned}$$

Transformations of W-matrices are transformations of no signalling processes.

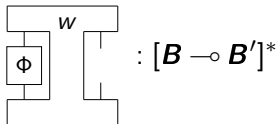




Dynamics of quantum causal structures

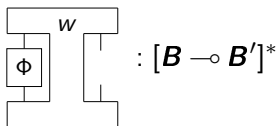
E. Castro-Ruiz, F. Giacomini, . Brukner - arXiv:1710.03139v2

Signalling for Process matrices

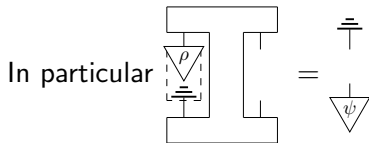


Dual for causal processes \mapsto factors

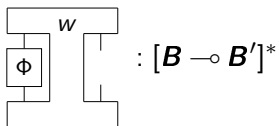
Signalling for Process matrices



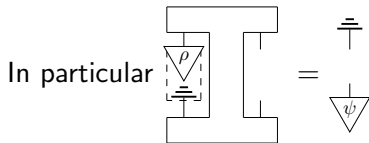
Dual for causal processes \mapsto factors



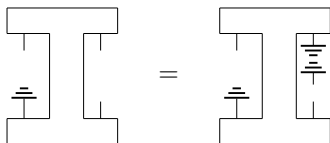
Signalling for Process matrices



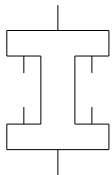
Dual for causal processes \mapsto factors



By enough causal states:

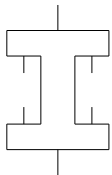


W-matrices with in/output



has type $[(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')] \multimap (\mathbf{C} \multimap \mathbf{C}')$

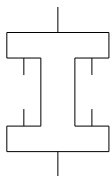
W-matrices with in/output



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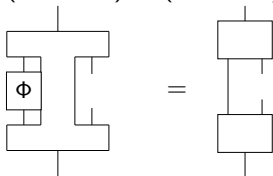
Plugging in a process $\Phi : \mathbf{A} \multimap \mathbf{A}'$ gives type
 $(\mathbf{B} \multimap \mathbf{B}') \multimap (\mathbf{C} \multimap \mathbf{C}') \cong \mathbf{C} \multimap (\mathbf{B} \multimap \mathbf{B}') \multimap \mathbf{C}'$

W-matrices with in/output



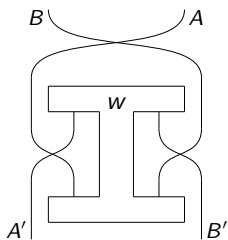
has type $[(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')] \multimap (\mathbf{C} \multimap \mathbf{C}')$

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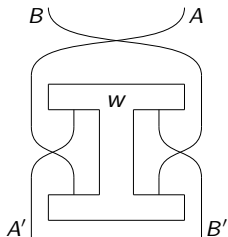
Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics

Ognyan Oreshkov - arXiv:1801.07594v2



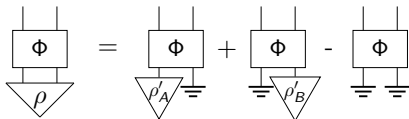
is no-signalling

Process matrices embed in no-signalling processes.



is no-signalling

Process matrices embed in no-signalling processes.
In QM we find this image as





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