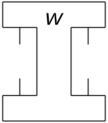
Process Matrices @ SYCO 2

Sander Uijlen

UNIVERSITY OF OXFORD
DEPARTMENT OF COMPUTER SCIENCE

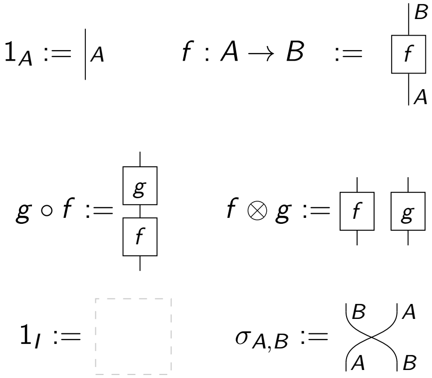
18 December - Glasgow

Say something about process matrices
on general grounds
using categorical semantics
for higher order processes.

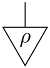


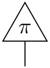
Process theories

Symmetric monoidal categories +
interpretation as systems and processes.



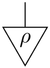
States and effects

 $: I \rightarrow A$ *state*

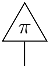
 $: A \rightarrow I$ *effect*

$\lambda : I \rightarrow I$ *scalars*

States and effects

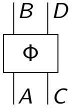


$: I \to A$ *state*

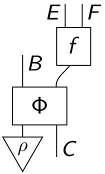


$: A \to I$ *effect*

$\lambda : I \to I$ *scalars*



$: A \otimes C \to B \otimes D$



$: C \to B \otimes E \otimes F$

Only connectivity matters!

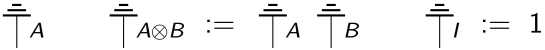
discarding

What else do we need to talk about caual orders?

discarding

What else do we need to talk about caual orders?

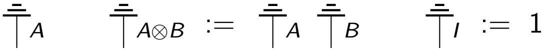
Consider a special family of *discarding* effects:



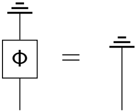
discarding

What else do we need to talk about caual orders?

Consider a special family of *discarding* effects:



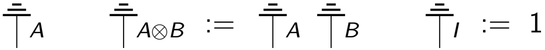
This enables us to say when a process is *causal*:



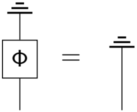
"If the outputs of a process are ignored, it doesn't matter which process happened."

discarding

What else do we need to talk about caual orders?
Consider a special family of *discarding* effects:



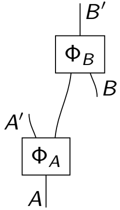
This enables us to say when a process is *causal*:



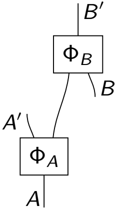
“If the outputs of a process are ignored, it doesn’t matter which process happened.”

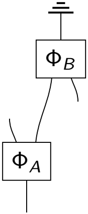
Consequence: A causal process only affects other processes which consume its outputs (i.e., those that lie in its *causal future*).

No signalling from the future

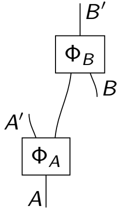


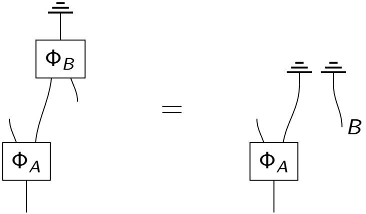
No signalling from the future



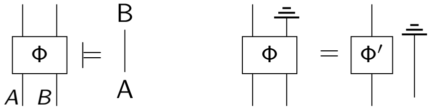


No signalling from the future





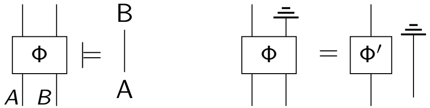
One-way signalling:



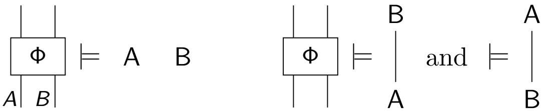
The output of A does not depend on B.

Causal orders

One-way signalling:

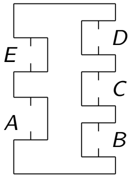


The output of A does not depend on B.
No-signalling



Dual processes

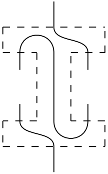
Causal order of dual process, e.g.,



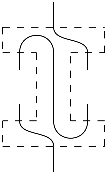
"What can they take in?"

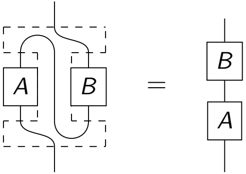
Dual processes can witness the causal order.

Higher Order Processes

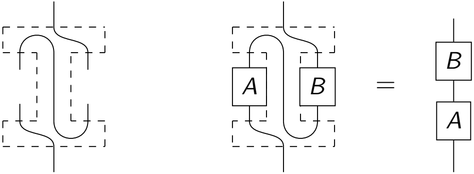


Higher Order Processes

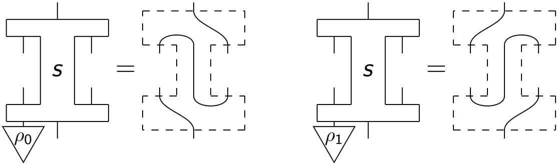




Higher Order Processes



Quantum switch:



Allows for 'coherent superpositions' of causal orders.

Compact closure

An easy way to get higher-order processes!

Compact closure

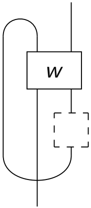
An easy way to get higher-order processes!

A way to 'bend wires'
(Choi-Jamiołkowski - Objects have duals A^*)

Satisfying

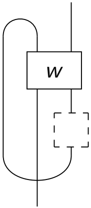
Problem

Take (causal) processes as inputs

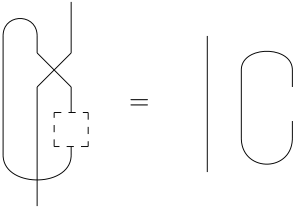


Problem

Take (causal) processes as inputs



...but this does not preserve causality:



Solution

Precausal category $\mathcal{C}$ $\mapsto$ $\mathrm{Caus}[\mathcal{C}]$

compact closed category
$+$ *4* ⏚ *axioms*

$*$*-autonomous category*
capturing 'logic of causality'

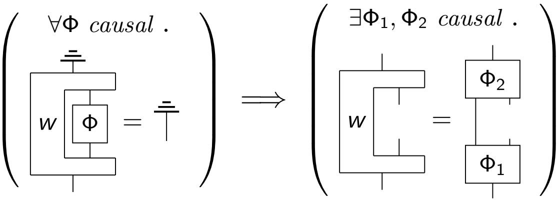
Solution

Precausal category $\mathcal{C}$	$\mapsto$	$\mathrm{Caus}[\mathcal{C}]$
compact closed category $+$ *4*  *axioms*		$*$*-autonomous category* *capturing 'logic of causality'*
Mat$(\mathbb{R}_+)$	$\mapsto$	HO stochastic maps
CPM	$\mapsto$	HO quantum channels

Input: Precausal Category

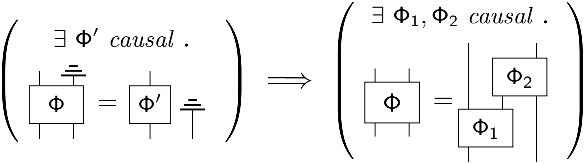
Compact closed category with $\bar{\top}$ + rules

- *Second-order causal* processes factorise:



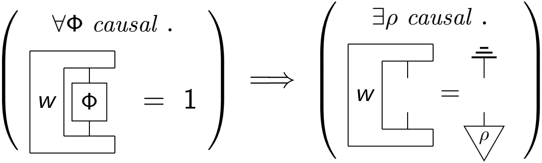
Equivalent to

- Causal *one-way signalling* processes factorise:



(Semicausal operations are semilocalizable)

- For all $w : A \otimes B^*$:



Output: Category of Higher Order Processes - $\mathrm{Caus}[\mathcal{C}]$

Build a new process theory describing 'causal' states:

Output: Category of Higher Order Processes - $\text{Caus}[\mathcal{C}]$

Build a new process theory describing 'causal' states:
New types:

$$\boldsymbol{A} := (A, c_{\boldsymbol{A}}) \text{ where } c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A))$$

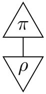
Output: Category of Higher Order Processes - $\mathrm{Caus}[\mathcal{C}]$

Build a new process theory describing 'causal' states:
New types:

$$\boldsymbol{A} := (A, c_{\boldsymbol{A}}) \text{ where } c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A))$$

With normalization and $c_{\boldsymbol{A}} = c_{\boldsymbol{A}}^{**}$

$$c^* \;:=\; \left\{ \pi : A^* \;\middle|\; \forall \rho \in c \;.\; \text{[diagram]} \;=\; 1 \right\}$$



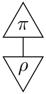
Output: Category of Higher Order Processes - $\mathrm{Caus}[\mathcal{C}]$

Build a new process theory describing 'causal' states:
New types:

$$\boldsymbol{A} := (A, c_{\boldsymbol{A}}) \text{ where } c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A))$$

With normalization and $c_{\boldsymbol{A}} = c_{\boldsymbol{A}}^{**}$

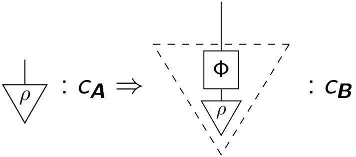
$$c^* \; := \; \left\{ \pi : A^* \;\middle|\; \forall \rho \in c \; . \right.$$



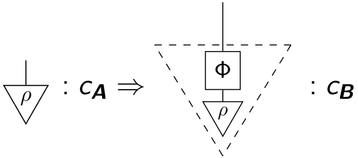
$$\left. = \; 1 \right\}$$

Types: causal states, causal processes, no-signalling processes...

Processes preserve the *generalized* causal states:



Processes preserve the *generalized* causal states:



- Causal states to causal states,
- Causal processes to a number (dual),
- No-signalling processes to causal processes,
- ...

$\mathrm{Caus}[\mathcal{C}]$ is ISOmix $*$-autonomous

We have a tensor, unit and duals

$$\boldsymbol{A} \otimes \boldsymbol{B} := (A \otimes B, (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}) \qquad \boldsymbol{I} := (I, \{1_I\}) \cong \boldsymbol{I}^*$$

$$\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}}^*)$$

$\mathrm{Caus}[\mathcal{C}]$ is ISOmix $*$-autonomous

We have a tensor, unit and duals

$$\boldsymbol{A} \otimes \boldsymbol{B} := (A \otimes B, (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}) \qquad \boldsymbol{I} := (I, \{1_I\}) \cong \boldsymbol{I}^*$$

$$\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}}^*)$$

...But no compact closure!

$$A \parr B := (A^* \otimes B^*)^* \neq A \otimes B$$

$\mathrm{Caus}[\mathcal{C}]$ is ISOmix $*$-autonomous

We have a tensor, unit and duals

$$\boldsymbol{A} \otimes \boldsymbol{B} := (A \otimes B, (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}) \qquad \boldsymbol{I} := (I, \{1_I\}) \cong \boldsymbol{I}^*$$

$$\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}}^*)$$

...But no compact closure!

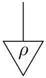
$$A \parr B := (A^* \otimes B^*)^* \neq A \otimes B$$

Define $A \multimap B := A^* \parr B$ giving internal hom

$$Hom(A \otimes B, C) \cong Hom(A, B \multimap C)$$

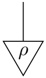
In particular $\qquad Hom(A, B) \cong Hom(I, A \multimap B)$

Examples

First order $(A, \{ \bar{\top}_A \}^*)$ 

Examples

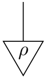
First order $(A, \{\overline{\top}_A\}^*)$



First order dual $(A^*, \{\overline{\top}_A\})$ $\overline{\top}_A$

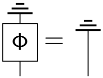
Examples

First order $(A, \{\top_A\}^*)$



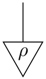
First order dual $(A^*, \{\top_A\})$ $\top_A$

Causal process $\boldsymbol{A \multimap B}$



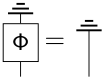
Examples

First order $(A, \{\bar{\top}_A\}^*)$

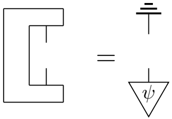


First order dual $(A^*, \{\bar{\top}_A\})$ $\bar{\top}_A$

Causal process $\boldsymbol{A} \multimap \boldsymbol{B}$

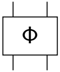


Dual of map $[\boldsymbol{A} \multimap \boldsymbol{B}]^*$



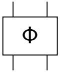
Types Capture Causal Orders

Bipartite process $(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$

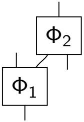


Types Capture Causal Orders

Bipartite process $(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$

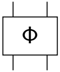


One-way signalling $\boldsymbol{A} \multimap ((\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}')$

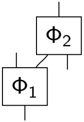


Types Capture Causal Orders

Bipartite process $(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$



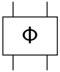
One-way signalling $\boldsymbol{A} \multimap ((\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}')$



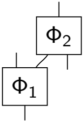
No signalling $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ both factorizations

Types Capture Causal Orders

Bipartite process $(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$

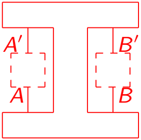


One-way signalling $\boldsymbol{A} \multimap ((\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}')$

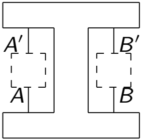


No signalling $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ both factorizations

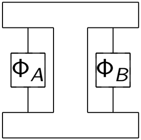
Dual-NS (W-matrix) $[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]^*$



Process Matrices



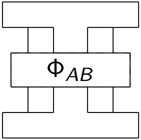
Satisfying for all causal Φ_A, Φ_B:



$= 1$

Process Matrices

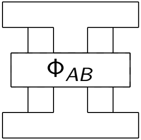
Equivalent to normalization on all no-signalling processes: for every no-signalling map $\Phi_{AB} : (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ we have



$= 1$

Process Matrices

Equivalent to normalization on all no-signalling processes: for every no-signalling map $\Phi_{AB} : (\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ we have



$= 1$

Definition

A process matrix is a process in the dual of no-signalling processes, i.e., a process of type $[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]^*$.


Y

Indefinite causal orders

- Most general way to obtain probabilities from no-signalling processes.

Indefinite causal orders

- Most general way to obtain probabilities from no-signalling processes.
- Interesting informational properties

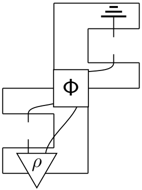
Indefinite causal orders

- Most general way to obtain probabilities from no-signalling processes.
- Interesting informational properties
- Can break causal bounds!
  Quantum correlations with no causal order
  Ognyan Oreshkov, Fabio Costa, Caslav Brukner - arXiv:1105.4464v3

Duals of one-way signalling processes

Process matrices compatible with a specific causal order

$$(\boldsymbol{A} \preceq \boldsymbol{B})^* := [\boldsymbol{A} \multimap ((\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}')]^*$$



Make a type which includes both duals $(\boldsymbol{A} \preceq \boldsymbol{B})^*$ and $(\boldsymbol{B} \preceq \boldsymbol{A})^*$

Proposition

The intersection of one-way signalling maps with $A \preceq B$ and one-way signalling maps $B \preceq A$ are the no-signalling maps.

$$\boldsymbol{A} \preceq \boldsymbol{B} \cap \boldsymbol{B} \preceq \boldsymbol{A} = \mathbf{NS}$$

Proposition

The intersection of one-way signalling maps with $A \preceq B$ and one-way signalling maps $B \preceq A$ are the no-signalling maps.

$$\boldsymbol{A} \preceq \boldsymbol{B} \cap \boldsymbol{B} \preceq \boldsymbol{A} = \mathbf{NS}$$

Theorem

$$[(\boldsymbol{A} \preceq \boldsymbol{B})^* \cup (\boldsymbol{B} \preceq \boldsymbol{A})^*]^{**} = \mathbf{NS}^*$$

Proposition

The intersection of one-way signalling maps with $A \preceq B$ and one-way signalling maps $B \preceq A$ are the no-signalling maps.

$$\boldsymbol{A} \preceq \boldsymbol{B} \cap \boldsymbol{B} \preceq \boldsymbol{A} = \mathbf{NS}$$

Theorem

$$[(\boldsymbol{A} \preceq \boldsymbol{B})^* \cup (\boldsymbol{B} \preceq \boldsymbol{A})^*]^{**} = \mathbf{NS}^*$$

Smallest type that contians boths duals are all process matrices.

What does it mean..?

- In QM and probability theory, the dubble dual is the positive affine closure.
- Every process matrix is an affine closure of duals of one-way signalling processes (combs).

What does it mean..?

- In QM and probability theory, the dubble dual is the positive affine closure.
- Every process matrix is an affine closure of duals of one-way signalling processes (combs).
- There are processes which are not *convex* combinations. (Switch, OCB, also in probability theory)

What does it mean..?

- In QM and probability theory, the dubble dual is the positive affine closure.
- Every process matrix is an affine closure of duals of one-way signalling processes (combs).
- There are processes which are not *convex* combinations. (Switch, OCB, also in probability theory)

$$\frac{\textit{defnite}}{\textit{indefinite}} \simeq \frac{\textit{seperable}}{\textit{entangled}}$$

Transformations of W-matrices

Type of such a transformation:

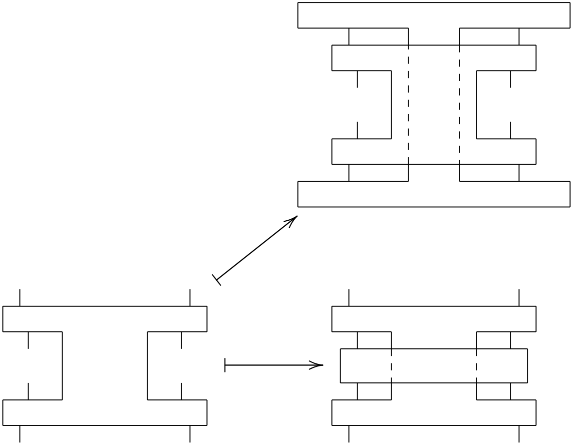
$$[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]^* \multimap [(\boldsymbol{C} \multimap \boldsymbol{C}') \otimes (\boldsymbol{D} \multimap \boldsymbol{D}')]^*$$

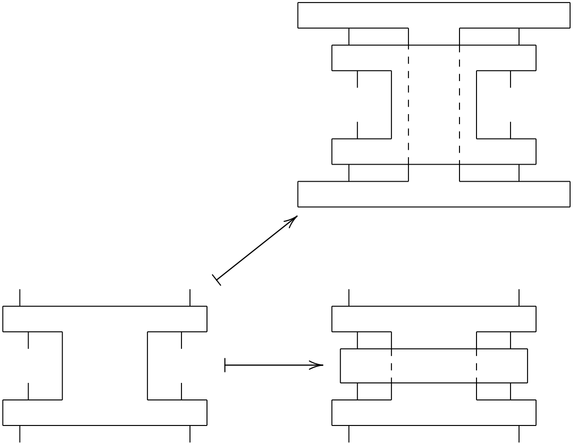
Transformations of W-matrices

Type of such a transformation:

$$[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]^* \multimap [(\boldsymbol{C} \multimap \boldsymbol{C}') \otimes (\boldsymbol{D} \multimap \boldsymbol{D}')]^*$$
$$\cong$$
$$[(\boldsymbol{C} \multimap \boldsymbol{C}') \otimes (\boldsymbol{D} \multimap \boldsymbol{D}')] \multimap [(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')]$$

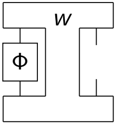
Transformations of W-matrices are transformations of no signalling processes.





Dynamics of quantum causal structures
E. Castro-Ruiz, F. Giacomini, . Brukner - arXiv:1710.03139v2

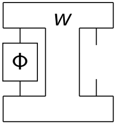
Signalling for Process matrices



$: [B \multimap B']^*$

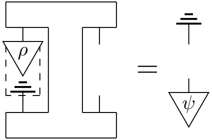
Dual for causal processes $\mapsto$ factors

Signalling for Process matrices

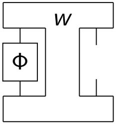


$: [\boldsymbol{B} \multimap \boldsymbol{B}']^*$

Dual for causal processes $\mapsto$ factors

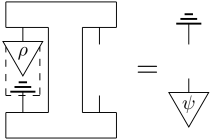
In particular 

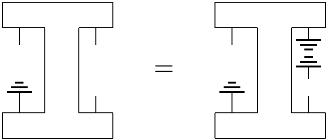
Signalling for Process matrices



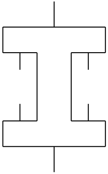
: $[\boldsymbol{B} \multimap \boldsymbol{B}']^*$

Dual for causal processes $\mapsto$ factors

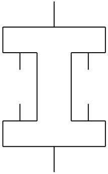
In particular 

By enough causal states: 

W-matrices with in/output

 has type $[(A \multimap A') \otimes (B \multimap B')] \multimap (C \multimap C')$

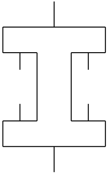
W-matrices with in/output



has type $[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')] \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$

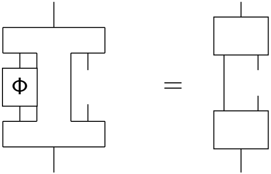
Plugging in a process $\Phi : \boldsymbol{A} \multimap \boldsymbol{A}'$ gives type
$(\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}') \cong \boldsymbol{C} \multimap (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$

W-matrices with in/output

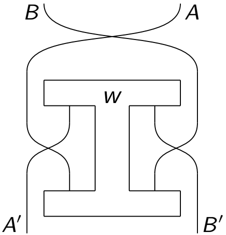


has type $[(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')] \multimap (\boldsymbol{C} \multimap \boldsymbol{C}')$

Plugging in a process $\Phi : \boldsymbol{A} \multimap \boldsymbol{A}'$ gives type
$(\boldsymbol{B} \multimap \boldsymbol{B}') \multimap (\boldsymbol{C} \multimap \boldsymbol{C}') \cong \boldsymbol{C} \multimap (\boldsymbol{B} \multimap \boldsymbol{B}') \multimap \boldsymbol{C}'$

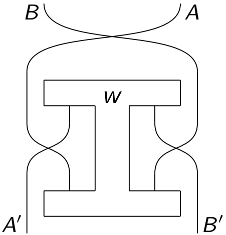


Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics
Ognyan Oreshkov - arXiv:1801.07594v2



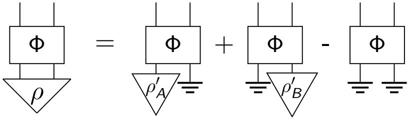
is no-signalling

Process matrices embed in no-signalling processes.



is no-signalling

Process matrices embed in no-signalling processes.
In QM we find this image as




ARE THERE ANY QUESTIONS? FEEL FREE TO ASK ANY-THING AT ALL.
Dilbert.com DilbertCartoonist@gmail.com
WHY DO GHOSTS HAVE CLOTHES?
9-22-11 ©2011 Scott Adams, Inc./Dist. by Universal Uclick
IF SOMEONE GIVES YOU A WEDGIE AT THE MOMENT YOU DIE, WILL YOU HAVE IT FOR ETERNITY?