# PyZX: Quantum circuit optimization using the ZX-calculus

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- So:
  Optimizing fault tolerant QC means optimizing T-count.

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Existing heuristics fall basically in two categories.

# Method 1: Commutation and Cancellation

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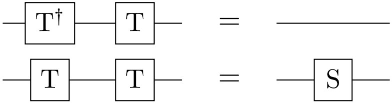

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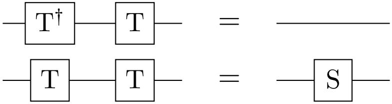


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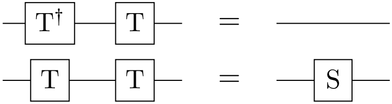


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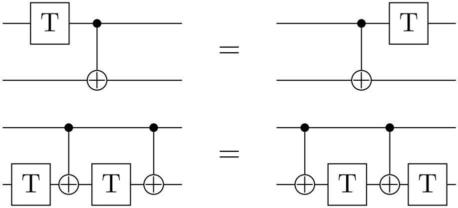

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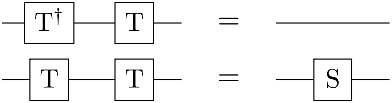


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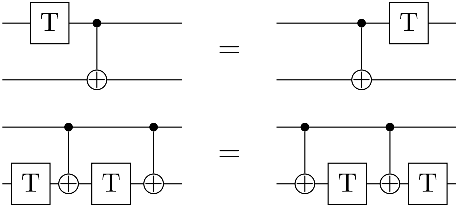


And loads more...

# Method 2: Phase Polynomials

Circuits built out of CNOT and T gates can be written as

$$U\,|\mathbf{x}\rangle = e^{i\frac{\pi}{4}g(\mathbf{x})}\,|f_1(\mathbf{x}),\ldots,f_n(\mathbf{x})\rangle$$

where $\mathbf{x} \in \mathbb{Z}_2^n$ is a binary vector,
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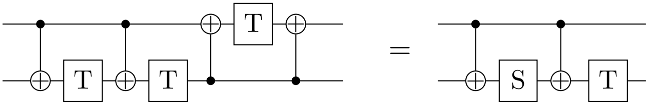


(but optimal T-count finding still seems to be in NP)

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Enter the ZX-calculus

# ZX-diagrams

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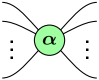
- Z-spiders 

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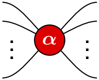
- X-spiders 

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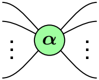
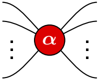
- Z-spiders  $:= \ |0\cdots0\rangle\langle0\cdots0| + e^{i\alpha}\,|1\cdots1\rangle\langle1\cdots1|$
- X-spiders  $:= \ |+\cdots+\rangle\langle+\cdots+| + e^{i\alpha}\,|-\cdots-\rangle\langle-\cdots-|$
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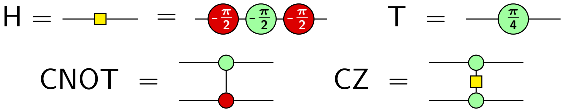

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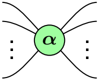
- Z-spiders 

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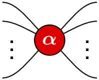
- X-spiders 

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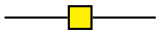
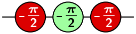
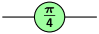
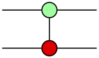
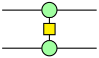
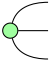
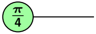
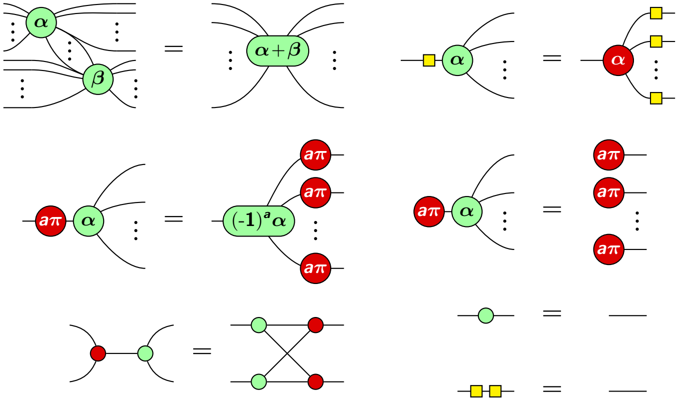
H =  = 

T = 


CNOT =  CZ = 

But also: GHZ =  T magic state = 

# ZX-calculus




$\alpha, \beta \in [0, 2\pi]$, $a \in \{0, 1\}$

# Circuit optimization with the ZX-calculus

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- Write your circuit as a ZX-diagram.
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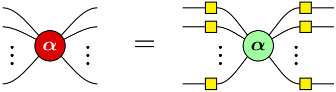
- Turn the resulting diagram back into a circuit.

# First steps: graph-like ZX-diagrams

- First turn all X-spiders into Z-spiders:



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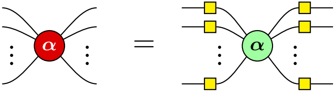
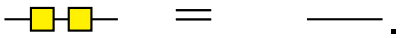



- Cancel all double Hadamards: 

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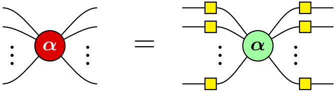
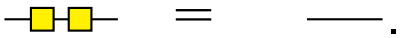



- Cancel all double Hadamards: 
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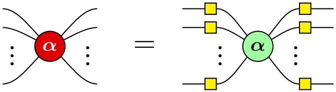
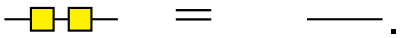



- Cancel all double Hadamards: 
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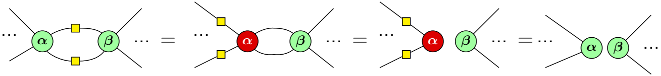

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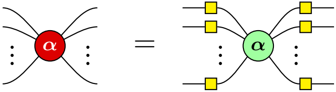
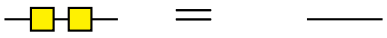



- Cancel all double Hadamards: .
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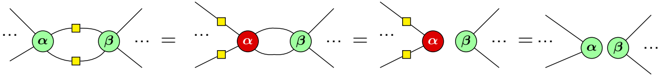
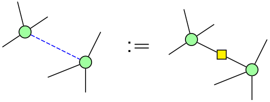
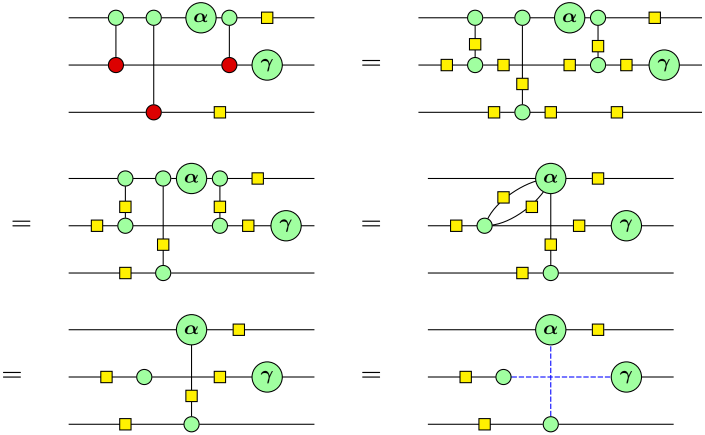


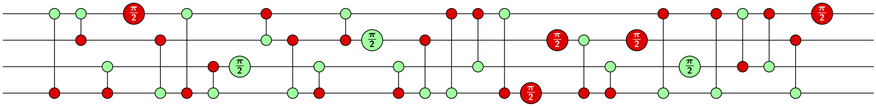

- Use new notation:



# Example



# More involved example

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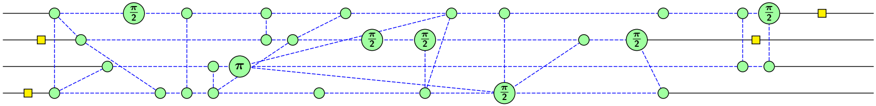

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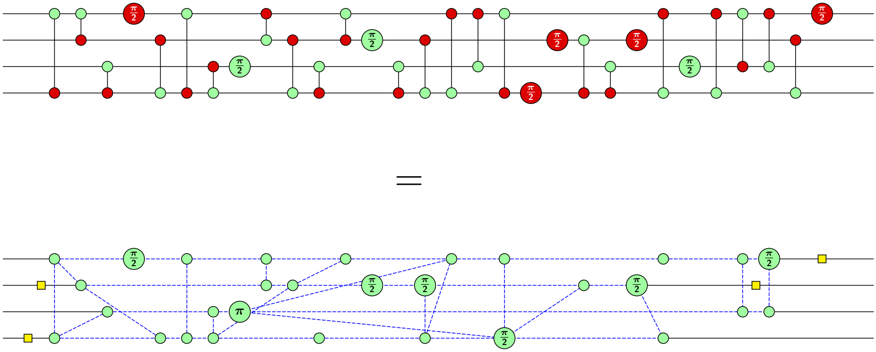

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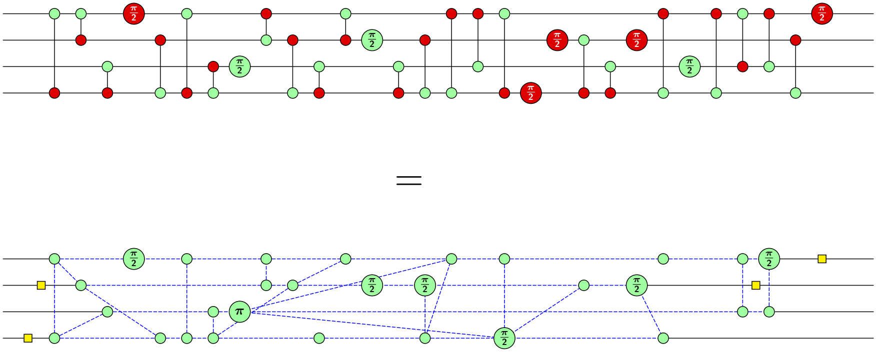
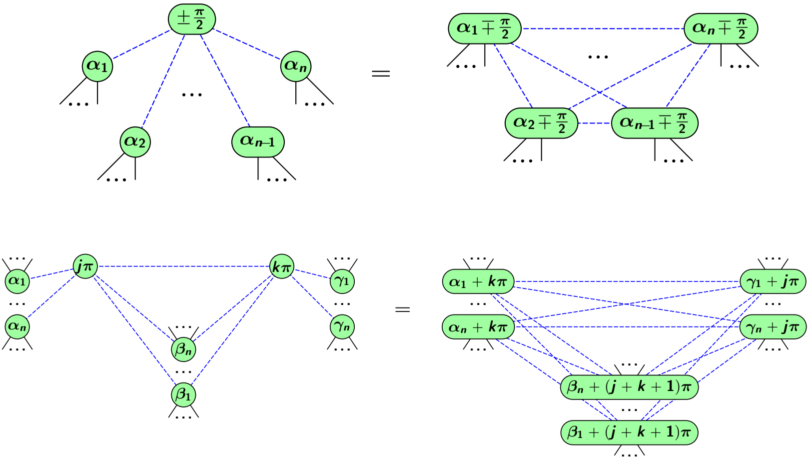



We call these diagrams *graph-like*.

To simplify these diagrams, we want to remove as many interior vertices as possible.

# Local complementation and pivoting



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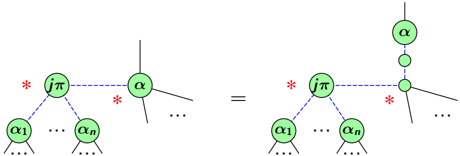

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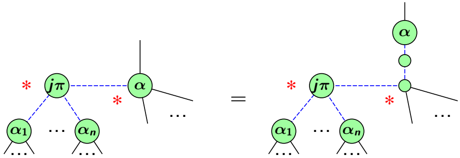
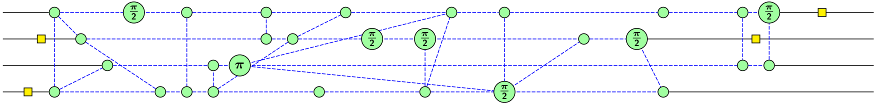



- If the original diagram was Clifford, then the simplified diagram has no internal spiders

# Clifford example

Recall the example Clifford diagram:



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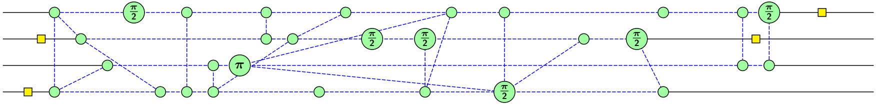
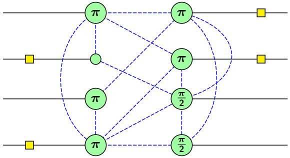
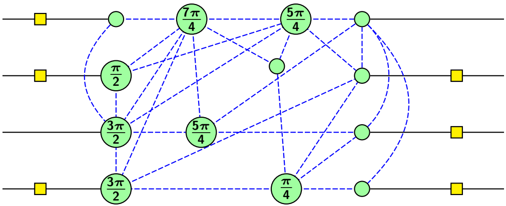



This can be reduced by the described procedure to:



# Clifford+T example



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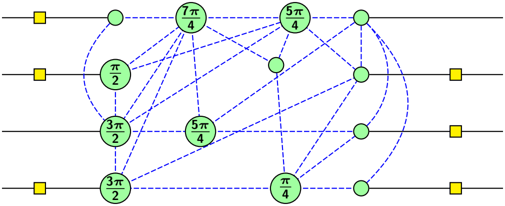


Question: How do we turn this into a circuit.

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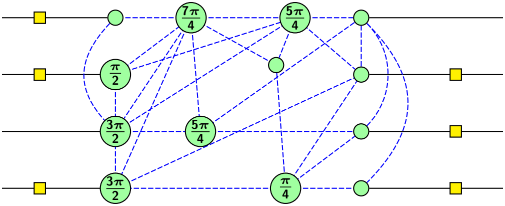


Question: How do we turn this into a circuit.
Answer: We use the fact that it has a *gFlow*.

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Local complementation and pivoting preserve gFlow

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Local complementation and pivoting preserve gFlow

### Theorem

There is an efficient procedure that transforms a ZX-diagram with a gFlow into a circuit.

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- We indeed have a circuit-to-circuit simplification procedure using the ZX-calculus.
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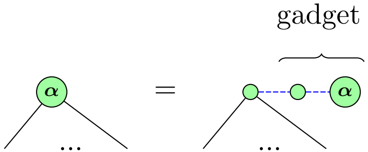
- But: T gates never get removed by lcomp and pivoting.
- So:
  To do significant T-count optimization, we need to do better.

# Gadgetization

We turn all non-Clifford spiders into *phase gadgets*:



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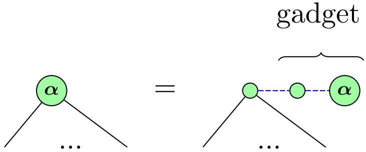
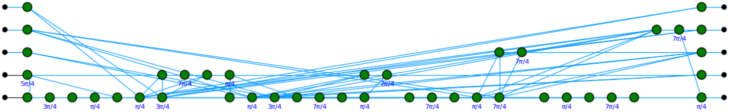



This makes the base of the gadget available for pivoting.

# Example: Gadgetization and pivoting

After the first round of simplifications:



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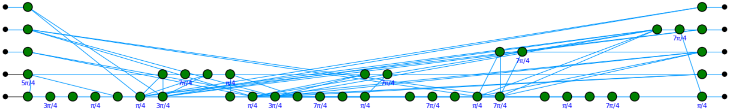
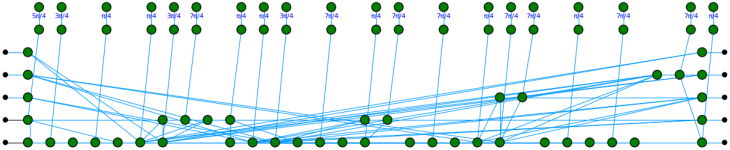
After the first round of simplifications:



After gadgetization:



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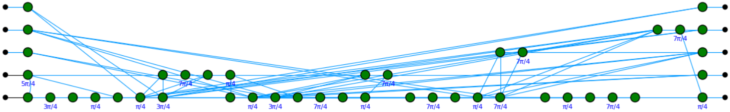
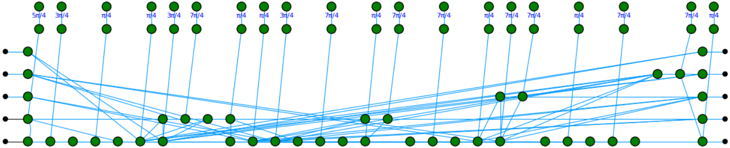
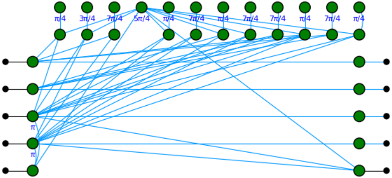


After gadgetization:

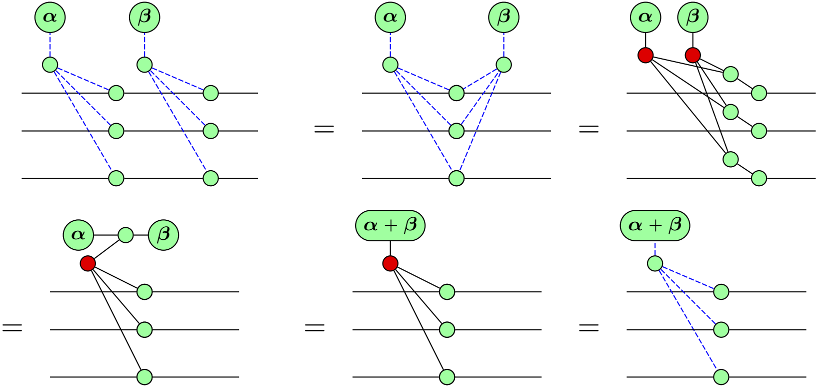


After pivots/lcomps:



# Final step: phase gadget fusion

Whenever phase gadgets have the same set of neighbours, they can fuse:



Spider fusion + Local complementation + Pivoting
+ Gadgetization + Gadget fusion
=
State-of-the-art T-count optimization

# Circuit extraction

Problem: We still need to get a circuit out of the diagram.

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The good news: We have 'heuristics' that always seem to work.
The bad news: We don't know why.

# Demonstration of PyZX

# Conclusion and future work

The Takeaway:

The ZX-calculus is very useful for
T-count optimization

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Open problems:

- Why does our circuit extraction work?

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Open problems:

- Why does our circuit extraction work?
- How to use phase-polynomial methods on ZX-diagrams?

# Conclusion and future work

The Takeaway:

The ZX-calculus is very useful for T-count optimization

Open problems:

- Why does our circuit extraction work?
- How to use phase-polynomial methods on ZX-diagrams?
- Is the ZH-calculus useful?

Thank you for your attention