On the Completeness of the ZX-calculus: from the Entire Qubit QM to Quantum Boolean Circuits

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Outline

Complete rules for the full qubit ZX-calculus

Complete rules for the Clifford+T ZX-calculus

Complete ZX rules for 2-qubit Clifford+T Circuits

Complete ZX rules for Quantum Boolean Circuits

Background

- The ZX-calculus proposed by Coecke and Duncan is a quantum diagram reasoning system presented by diagrams as generators and rewriting rules of diagrams as relations.

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- Completeness of the ZX-calculus means any equality that can be derived using matrices can also be derived by rewriting ZX diagrams.
- There are quite a few completeness results for the ZX-calculus. We only introduce our own results here.

Generators of the ZX-calculus

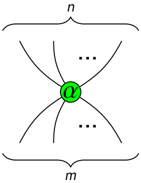
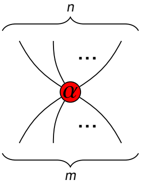
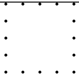
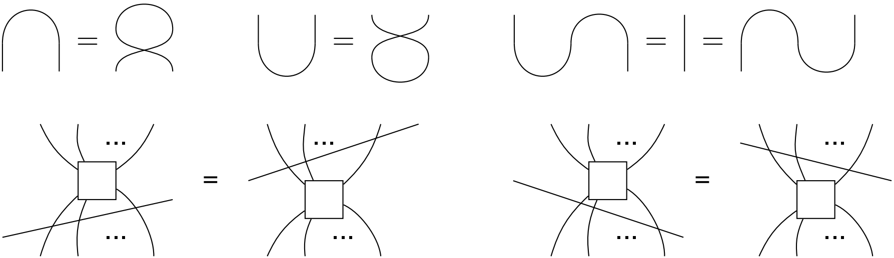
$R_{Z,\alpha}^{(n,m)} : n \to m$		$R_{X,\alpha}^{(n,m)} : n \to m$	
$H : 1 \to 1$		$\sigma : 2 \to 2$	
$\mathbb{I} : 1 \to 1$		$e : 0 \to 0$	
$C_a : 0 \to 2$		$C_u : 2 \to 0$	
$L : 1 \to 1$		$T : 1 \to 1$	

Table: Generators of qubit ZX-calculus

where $m, n \in \mathbb{N}$, $\alpha \in [0, 2\pi)$, $\lambda \geq 0$, and e represents an empty diagram.

Structural rules of the ZX-calculus



Non-structural rules of the ZX-calculus

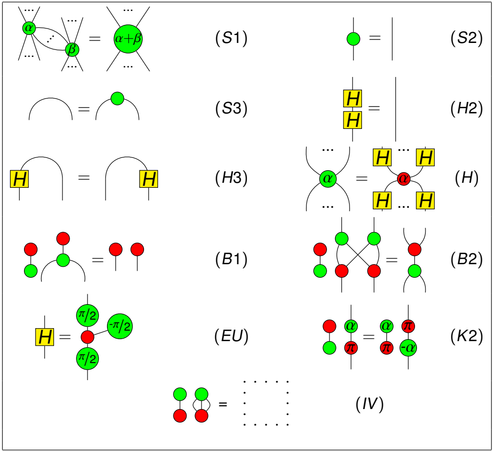


Figure: Non-structural ZX-calculus rules, where $\alpha, \beta \in [0,\ 2\pi)$.

Note that all the rules enumerated in Figures 1 still hold when they are flipped upside-down. Due to the rule (H) and (H2), the rules in Figure 1 have a property that they still hold when the colours green and red swapped.

Non-structural rules of the ZX-calculus

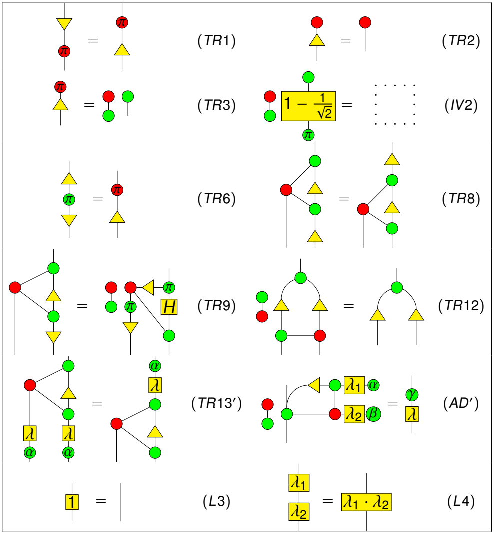
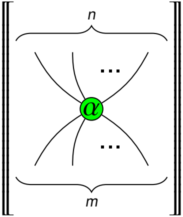
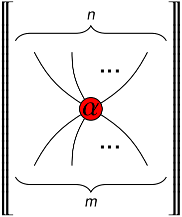


Figure: Extended ZX-calculus rules, where $\lambda, \lambda_1, \lambda_2 \geq 0, \alpha, \beta, \gamma \in [0,\ 2\pi)$; in (AD′), $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$.The upside-down version of these rules still hold.

Standard interpretation of the ZX-calculus in **Qubit**

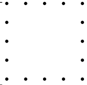


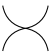
$$= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n},$$



$$= |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\alpha} |-\rangle^{\otimes m} \langle -|^{\otimes n},$$

Standard interpretation of the ZX-calculus

$[\![$  $]\!] = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $[\![$  $]\!] = 1$, $[\![$  $]\!] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$,

$[\![$  $]\!] = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$, $[\![$  $]\!] = \begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}$, $[\![$  $]\!] = \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}$,

$[\![$  $]\!] = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $[\![$  $]\!] = \begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}$.

$$[\![D_1 \otimes D_2]\!] = [\![D_1]\!] \otimes [\![D_2]\!], \quad [\![D_1 \circ D_2]\!] = [\![D_1]\!] \circ [\![D_2]\!],$$

ZX-calculus as a quantum diagram reasoning system

- Definition

  The ZX-calculus is called sound if for any two diagrams D_1 and D_2, $ZX \vdash D_1 = D_2$ must imply that $[\![D_1]\!] = [\![D_2]\!]$.

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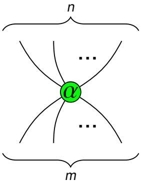
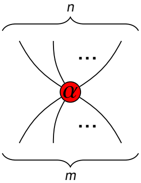
  The ZX-calculus is called universal if for any linear map L, there must exist a diagram D in the ZX-calculus such that $[\![D]\!] = L$.
- Definition
  The ZX-calculus is called complete if for any two diagrams D_1 and D_2, $[\![D_1]\!] = [\![D_2]\!]$ must imply that $ZX \vdash D_1 = D_2$.

Completeness of the ZX-calculus

Theorem (Ng & Wang)

This version of ZX-calculus is complete for the entire pure qubit quantum mechanics.

Generators of the Clifford+T ZX-calculus

$R_{Z,\alpha}^{(n,m)} : n \to m$		$R_{X,\alpha}^{(n,m)} : n \to m$	
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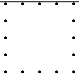
$\mathbb{I} : 1 \to 1$		$e : 0 \to 0$	
$C_a : 0 \to 2$		$C_u : 2 \to 0$	
$L : 1 \to 1$		$T : 1 \to 1$	

Table: Generators of the Clifford+T ZX-calculus

where $m, n \in \mathbb{N}$, $\alpha \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}, 0 \leqslant \lambda \in \mathbb{Z}[\frac{1}{2}]$.

Non-structural rules of the Clifford+T ZX-calculus

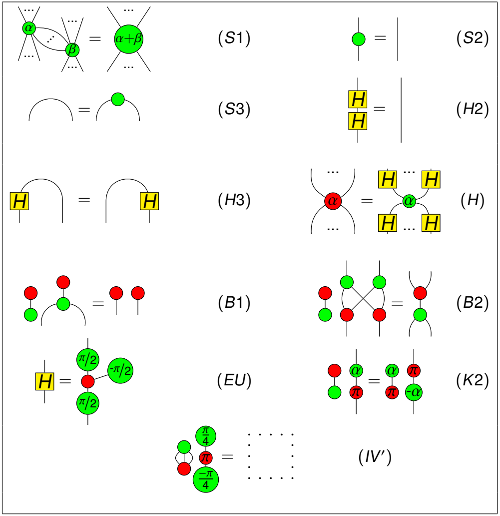


Figure: Traditional-style ZX_{C+T}-calculus rules, where $\alpha, \beta \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}$.
The upside-down version and colour swapped version of these rules still hold.

Non-structural rules of the Clifford+T ZX-calculus

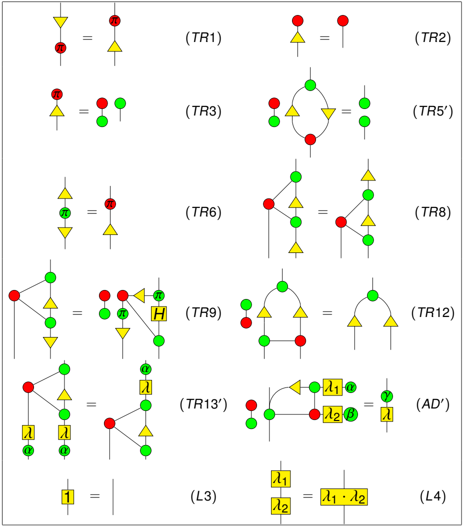


Figure: ZX_{C+T}-calculus rules with triangle and λ box, where $0 \leqslant \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}[\frac{1}{2}], \alpha \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}, \alpha \equiv \beta \equiv \gamma \ (mod\ \pi)$ in (AD′). The upside-down version of these rules still hold.

Completeness of the Clifford+T ZX-calculus

Theorem (Ng & Wang)

This version of ZX-calculus is complete for the Clifford+T quantum mechanics.

Efficiency of complete ZX rules

- Given a complete set of ZX rules, we may need exponentially many steps in some particular situation when applying these rules.

Efficiency of complete ZX rules

- Given a complete set of ZX rules, we may need exponentially many steps in some particular situation when applying these rules.
- How to single out useful ZX rules is essential for application of the ZX-calculus.

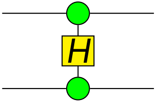
Basic quantum gates in ZX

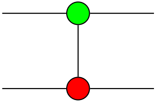
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto$$ 

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto$$ 

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \mapsto$$ 

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \mapsto$$ 

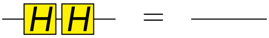
$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mapsto$$ 

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mapsto$$ 

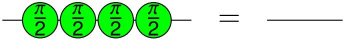
Complete Circuit Templates

Theorem (Selinger and Bian, 2015)

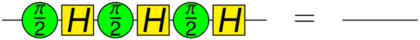
The following 17 equations are complete for 2-qubit Clifford+T circuits:



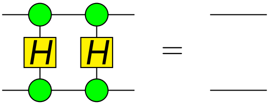
(1)



(2)



(3)



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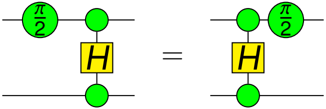
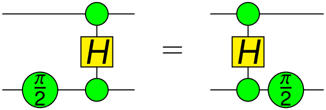
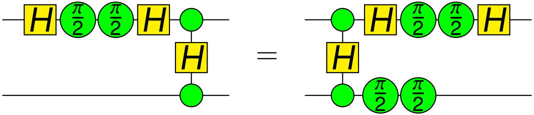


(5)

Complete Circuit Templates



(6)



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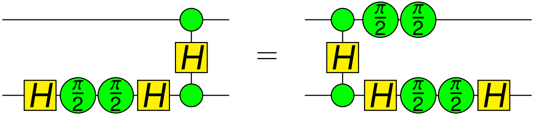

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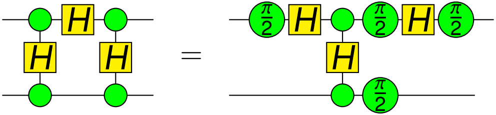
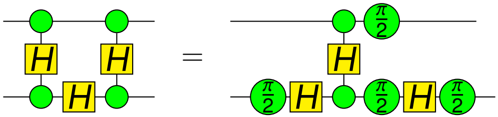
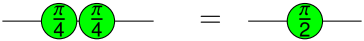


(9)

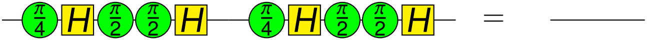
Complete Circuit Templates



(10)



(11)



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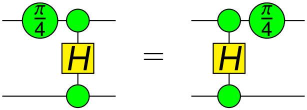

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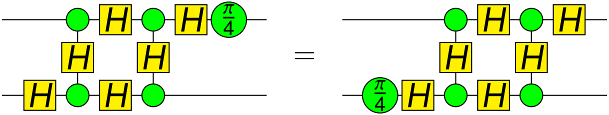
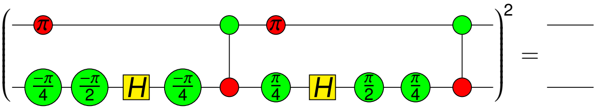


(14)

Complete Circuit Templates



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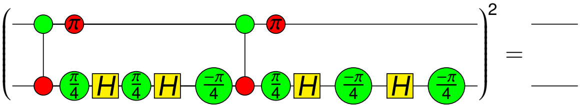

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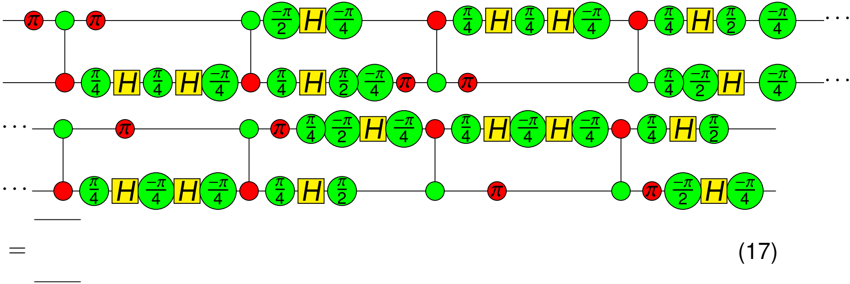


(17)

Rules of ZX-calculus for 2-qubit Clifford+T Quantum Circuits

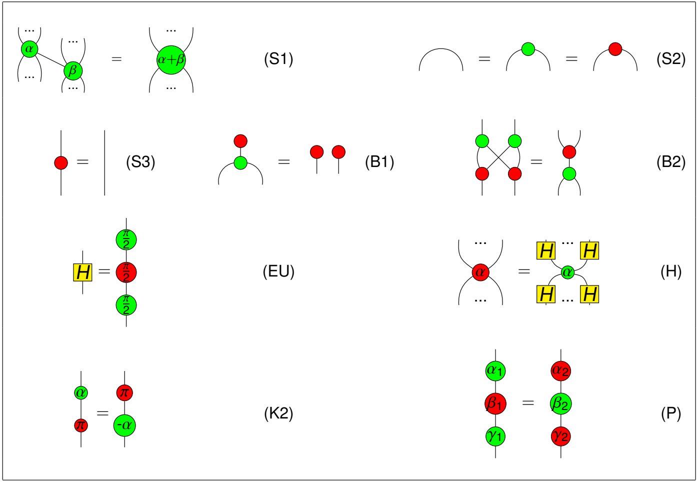
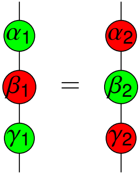


Figure: where $\alpha, \beta \in [0, 2\pi)$. The exact formula for the rule (P) is given in (18), but we only need to know that the (P) rule hold and to use the property that if $\alpha_1 = \gamma_1$, then $\alpha_2 = \gamma_2$, and if $\alpha_1 = -\gamma_1$, then $\alpha_2 = \pi + \gamma_2$.

Details of the (P) rule

Theorem

For $\alpha_1, \beta_1, \gamma_1 \in (0,\ 2\pi)$ *we have:*



with

$$\begin{cases} \alpha_2 = \arg z + \arg z_1 \\ \beta_2 = 2\arg(|\frac{z}{z_1}| + i) \\ \gamma_2 = \arg z - \arg z_1 \end{cases} \tag{18}$$

where:

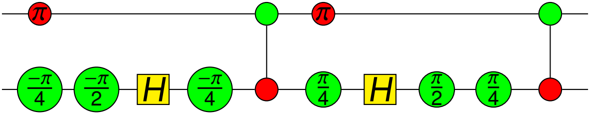
$$z = \cos\frac{\beta_1}{2}\cos\frac{\alpha_1+\gamma_1}{2} + i\sin\frac{\beta_1}{2}\cos\frac{\alpha_1-\gamma_1}{2}$$
$$z_1 = \cos\frac{\beta_1}{2}\sin\frac{\alpha_1+\gamma_1}{2} - i\sin\frac{\beta_1}{2}\sin\frac{\alpha_1-\gamma_1}{2}$$

So if $\alpha_1 = \gamma_1$*, then* $\alpha_2 = \gamma_2$*, and if* $\alpha_1 = -\gamma_1$*, then* $\alpha_2 = \pi + \gamma_2$*.*

Example of Application of (P) Rule

Lemma

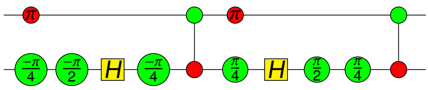
Let $A =$



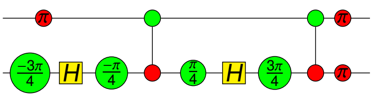
then $A^2 = I$.

Example of Application of (P) Rule

First we have $A =$



$\overset{B1,S1}{=}$



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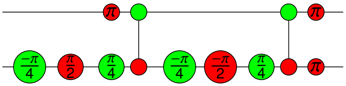
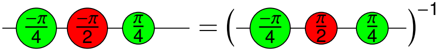


By the rule (P), we can assume that

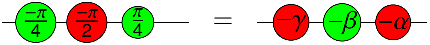
(19)

Example of Application of (P) Rule

Since $e^{i\frac{-\pi}{4}} e^{i\frac{\pi}{4}} = 1$, we could let $\gamma = \alpha + \pi$. Also note that

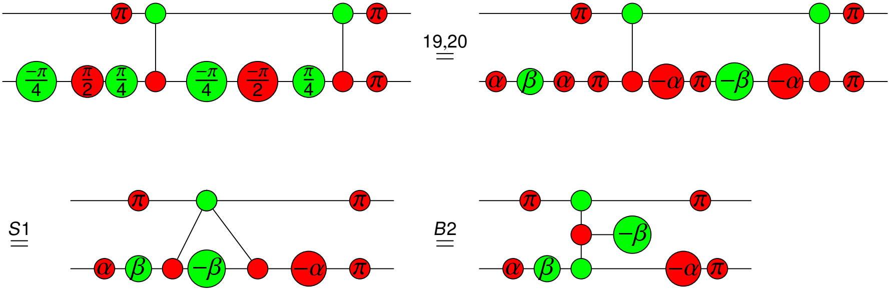


Thus



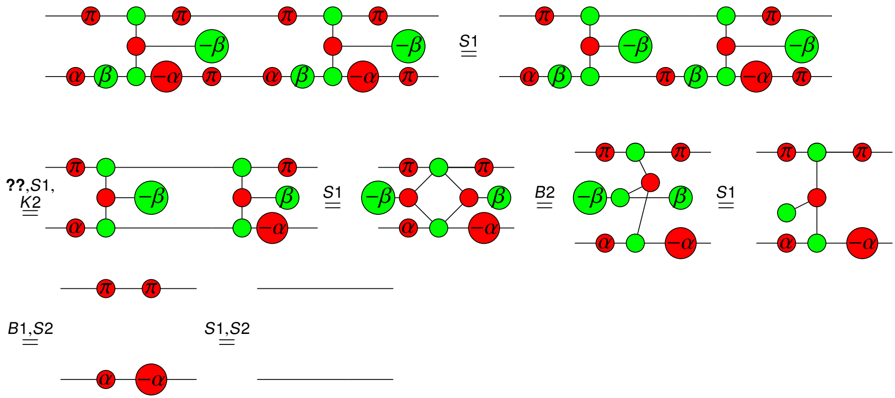
(20)

Therefore, $A =$



Example of Application of (P) Rule

Finally, $A^2 =$



Completeness of the ZX-calculus

Theorem (Coecke & Wang)

This version of ZX-calculus is complete for the 2-qubit quantum circuits.

Quantum Boolean Circuits

- A Control-NOT gate (CNOT) gate (also called n-bit Toffoli gate) is denoted by $[t, C]$, where t is an integer and C is a finite set of integers ($t \notin C$). $|x_t\rangle$ is called a target bit and $|x_k\rangle$ is called a control bit if $k \in C$.

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- A quantum Boolean circuit of size M over qubits $|x_1\rangle, ..., |x_N\rangle$ is a sequence of CNOT gates $[t_1, C_1] \cdots [t_i, C_i] \cdots [t_M, C_M]$ where $1 \leq t_i \leq N$ and $C_i \subseteq \{1, ..., N\}$.

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- A Control-NOT gate (CNOT) gate (also called n-bit Toffoli gate) is denoted by $[t, C]$, where t is an integer and C is a finite set of integers $(t \notin C)$. $|x_t\rangle$ is called a target bit and $|x_k\rangle$ is called a control bit if $k \in C$.
- A quantum Boolean circuit of size M over qubits $|x_1\rangle, ..., |x_N\rangle$ is a sequence of CNOT gates $[t_1, C_1] \cdots [t_i, C_i] \cdots [t_M, C_M]$ where $1 \leq t_i \leq N$ and $C_i \subseteq \{1, ..., N\}$.
- A quantum Boolean circuit is said to be proper to compute a Boolean function $f(x_1, \cdots, x_n)$ iff (i) The initial state $S_0 = |a_1\rangle |a_2\rangle \cdots \left|a_{n+1}\right\rangle |0\rangle \cdots |0\rangle$, (ii) The final state $S_M = |a_1\rangle |a_2\rangle \cdots \left|a_{n+1} \oplus f(x_1, \cdots, x_n)\right\rangle |0\rangle \cdots |0\rangle$.

Iwama, K., Kambayashi, Y., Yamashita, S.: Transformation Rules for Designing CNOT-based Quantum Circuits., 2002.

A Picture of Quantum Boolean Circuit

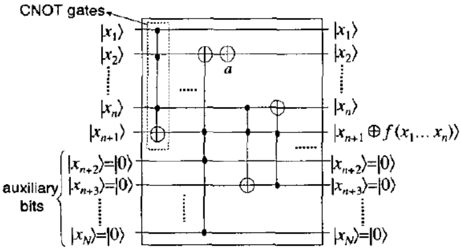


Figure 1: A Quantum Boolean Circuit

Iwama, K., Kambayashi, Y., Yamashita, S.: Transformation Rules for Designing CNOT-based Quantum Circuits., 2002.

Transformation Rules in Quantum Boolean Circuits

Letting ε represent the 'identity' gate and $\Longleftrightarrow$ represent a transformation, the transformation rule set is given as follows:

(1) $[t_1, C_1] \Longleftrightarrow \varepsilon$
(2) $[t_1, C_1] \cdot [t_2, C_2] \Longleftrightarrow [t_2, C_2] \cdot [t_1, C_1]$

(3) $[t_1, C_1] \cdot [t_2, C_2] \Longleftrightarrow [t_2, C_2] \cdot [t_1, C_1] \cdot [t_1, C_1 \cup C_2 - \{t_2\}]$ if $t_1 \notin C_2$ and $t_2 \in C_1$
(4) $[t_1, C_1] \cdot [t_2, C_2] \Longleftrightarrow [t_2, C_1 \cup C_2 - \{t_1\}] \cdot [t_2, C_2] \cdot [t_1, C_1]$ if $t_1 \in C_2$ and $t_2 \notin C_1$

(5) $[t_1, \{c_1\}] \cdot [t_2, C_2 \cup \{c_1\}] \Longleftrightarrow [t_1, \{c_1\}] \cdot [t_1, \{c_1\}] \cdot [t_2, C_2 \cup \{t_1\}]$ if $t_1 > n + 1$ and there is no CNOT_{t_1} before $[t_1, \{c_1\}]$

(6) $[t, C] \Longleftrightarrow \varepsilon$ if there is an integer i such that $i \in C$, $i > n + 1$, and there is no CNOT_i before $[t, C]$

Iwama, K., Kambayashi, Y., Yamashita, S.: Transformation Rules for Designing CNOT-based Quantum Circuits., 2002.

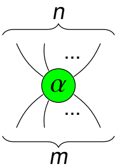
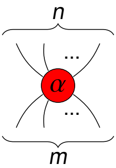
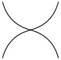
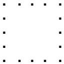
Completeness for Quantum Boolean Circuits

Theorem

Let S_1 and S_2 be any equivalent proper quantum Boolean circuits. Then there exists a sequence of transformation rules which transforms S_1 to S_2.

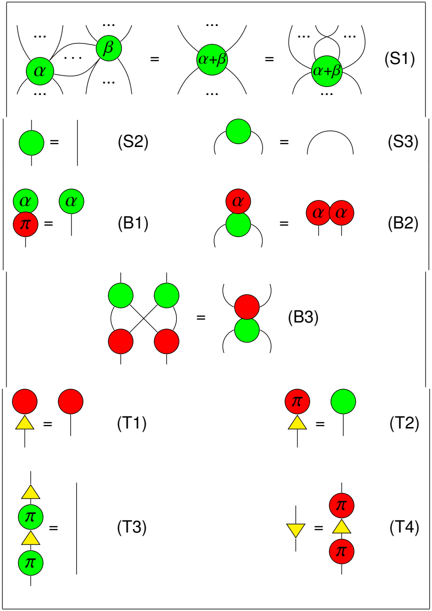
Iwama, K., Kambayashi, Y., Yamashita, S.: Transformation Rules for Designing CNOT-based Quantum Circuits., 2002.

Generators of the ZX-calculus for Quantum Boolean Circuits

$Z_\alpha^{(n,m)} : n \to m$		$X_\alpha^{(n,m)} : n \to m$	
$T : 1 \to 1$		$\mathbb{I} : 1 \to 1$	
$C_u : 2 \to 0$		$C_a : 0 \to 2$	
$\sigma : 2 \to 2$		$e : 0 \to 0$	

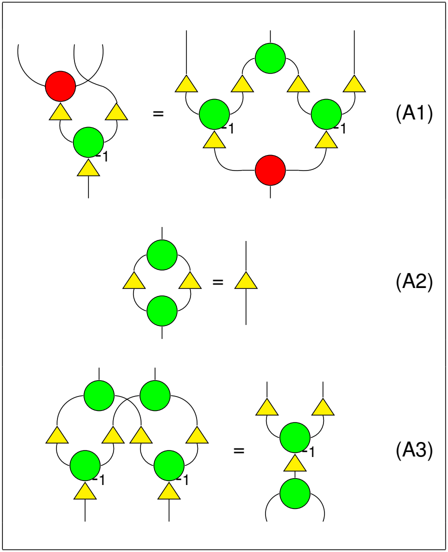
where $m, n \in \mathbb{N}$, $\alpha \in \{0, \pi\}$.

ZX-calculus Rules for Quantum Boolean Circuits



where $\alpha \in \{0, \pi\}$.

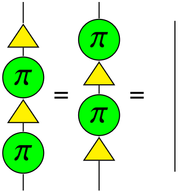
ZX-calculus Rules for Quantum Boolean Circuits



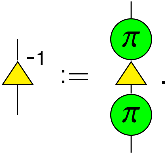
where $\alpha \in \{0, \pi\}$.

Inverse of the Triangle

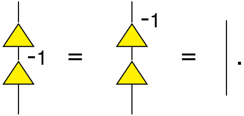
Note that, in combination, rules (S1) and (T3) imply that



so we may define the inverse of the ‘triangle’ diagram $T : 1 \to 1$ as

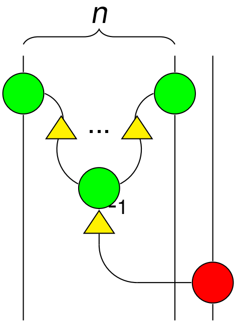


Then



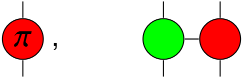
Toffoli Gate in ZX

Now the ZX-diagram for a Toffoli (CNOT) gate is simply



.

Especially in the $n = 0$ and $n = 1$ cases, this representation reduces to the NOT gate and the standard CNOT gate, as expected:



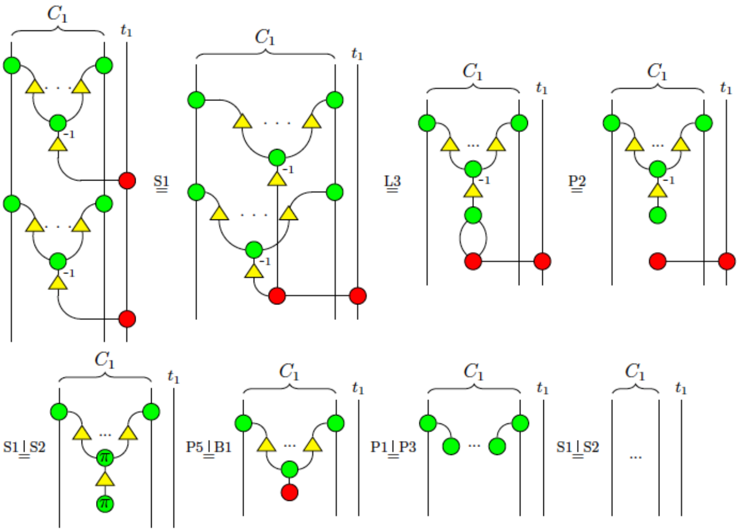
Completeness of the ZX-calculus for Quantum Boolean Circuits

Theorem (Coecke, Munson, Wang)

All the rules from (1) to (6) can be derived from the above ZX rules for quantum Boolean circuits, i.e., the ZX-calculus is complete for the quantum Boolean circuits.

Example of Derivation

(1) $[t_1, C_1] \iff \varepsilon$



Further work

- Generalise the completeness result of the ZX-calculus from qubit to qudit for arbitrary dimension d.

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- Achieve a complete axiomatization of the ZX-calculus with mixed dimensions.

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- Generalise the completeness result of the ZX-calculus from qubit to qudit for arbitrary dimension d.
- Achieve a complete axiomatization of the ZX-calculus with mixed dimensions.
- Efficient ZX rules for Benchmark quantum circuits.

Thank you!