Change Actions

Models of Generalised Differentiation

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Incremental computation with derivatives

- Objective: compute the value of an (expensive) function f
- Input x changes over time: $x_1, x_2, \ldots$
- *How to update the value of $f(x)$ as x_i changes?*

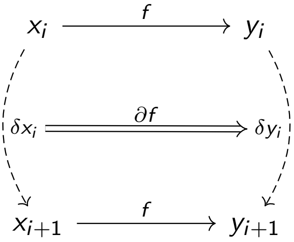
Incremental computation with derivatives

- Interpret the x_i as applying successive "updates" δx_i to an initial value x_1:

$$x_2 = x_1 \oplus \delta x_1$$
$$x_3 = x_2 \oplus \delta x_2 \ldots$$

- Find δy_i such that:

$$f(x_2) = f(x_1 \oplus \delta x_1) = f(x_1) \oplus \delta y_1$$
$$f(x_3) = f(x_2 \oplus \delta x_2) = f(x_2) \oplus \delta y_2 \ldots$$



Change action

A **change action** $\overline{A}$ (in a Cartesian category $\mathbf{C}$) is a tuple $(A, \Delta A, \oplus, +, 0)$ such that:

- $(\Delta A, +, 0)$ is a monoid
- $\oplus : A \times \Delta A \to A$ is an action of ΔA on A, i.e.:
  1. $a \oplus 0 = a$
  2. $a \oplus (\delta a + \delta b) = (a \oplus \delta a) \oplus \delta b$

Morphisms between change actions

Change actions

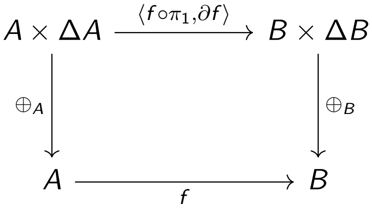
Given change actions $\overline{A}, \overline{B}$ and a map $f : A \to B$, a **derivative** for f is a function $\partial f : A \times \Delta A \to \Delta B$ such that:

- $f(a \oplus \delta a) = f(a) \oplus \partial f(a, \delta a)$
- $\partial f(a, 0_A) = 0_B$
- $\partial f(a, \delta a + \delta b) = \partial f(a, \delta a) + \partial f(a \oplus \delta a, \delta b)$

What we *don't* require:

- Linearity
- Uniqueness!

Diagramatically:

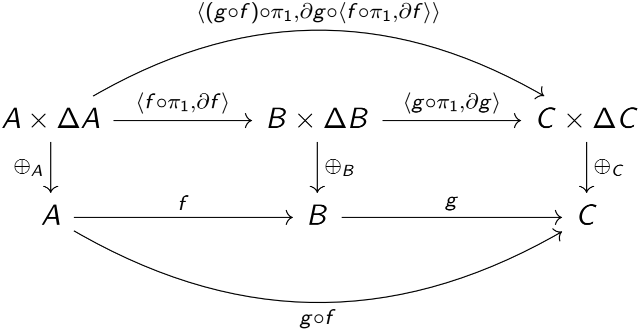


Condition 1 essentially says: $\oplus$ is a natural transformation!

The chain rule

Theorem

Given $f : A \to B, g : B \to C$ differentiable maps with derivatives $\partial f, \partial g$, then $\partial g(f(a), \partial f(a, \delta a))$ is a derivative for $g \circ f$



A category of change actions

The category $\mathrm{CAct}(\mathbf{C})$

Given a Cartesian category $\mathbf{C}$, we define the category $\mathrm{CAct}(\mathbf{C})$ as follows:

- Objects of $\mathrm{CAct}(\mathbf{C})$: all $\mathbf{C}$-change actions $\overline{A} = (A, \Delta A, \oplus, +, 0)$
- Morphisms $\overline{f} : \overline{A} \to \overline{B}$: pairs $(f, \partial f)$ of $\mathbf{C}$-map f and derivative for f.
- Identities: $\overline{\mathrm{Id}} = (\mathrm{Id}, \pi_2)$
- Composition: chain rule!

Lemma

The above induces an endofunctor $\mathrm{CAct} : \mathbf{Cat}_\times \to \mathbf{Cat}_\times$

Products in $\mathrm{CAct}(\mathbf{C})$

Product of change actions

Given change actions $\overline{A}, \overline{B}$, their product $\overline{A} \times \overline{B}$ is given by:

$$\begin{aligned} \overline{A} \times \overline{B} &= (A \times B, \Delta A \times \Delta B, \oplus_\times, +_\times, 0_\times) \\ (a, b) \oplus_\times (\delta a, \delta b) &= (a \oplus_A \delta a, b \oplus_B \delta b) \\ (\delta a_1, \delta b_1) +_\times (\delta a_2, \delta b_2) &= (\delta a_1 +_A \delta a_2, \delta b_1 +_B \delta b_2) \\ 0_\times &= (0_A, 0_B) \end{aligned}$$

Terminal object: $\overline{\top} = (\top, \top, \ldots)$

Coproducts in $\mathrm{CAct}(\mathbf{C})$

Whenever **C** has (distributive) coproducts, so does $\mathrm{CAct}(\mathbf{C})$!

Coproduct of change actions

Given difference algebras $\overline{A}, \overline{B}$, their coproduct difference algebra $\overline{A} + \overline{B}$ is given by:

$$\begin{aligned}
\overline{A} + \overline{B} &= (A + B, \Delta A \times \Delta B, \oplus_+, +_+, 0_+) \\
a \oplus_+ (\delta a, \delta b) &= a \oplus_A \delta a \\
b \oplus_+ (\delta a, \delta b) &= b \oplus_B \delta b \\
(\delta a_1, \delta b_1) +_+ (\delta a_2, \delta b_2) &= (da_1 +_A \delta a_2, \delta b_1 +_B \delta b_2) \\
0_+ &= (0_A, 0_B)
\end{aligned}$$

Initial object: $\overline{\bot} = (\bot, \top, \ldots)$

(Corollary: the derivative of a constant map is 0!)

Higher-order derivatives

- All derivatives so far: first-order!
  - No $\partial\partial f$
- How to get higher order derivatives?
  - Idea: make ΔA a change action

Change action models

A change action model on a Cartesian category **C** is a section $\alpha : \mathbf{C} \to \mathrm{CAct}(\mathbf{C})$ of the obvious forgetful functor ϵ, that is, α is a product-preserving functor from **C** into $\mathrm{CAct}(\mathbf{C})$ such that $\varepsilon \circ \alpha = \mathrm{Id}$

Notation: when A is a **C**-object, we use $\Delta A, \oplus, +, 0$ for those in $\alpha(A)$ - same for f.

Higher-order derivatives

Some consequences of the previous definition:

- Higher-order derivatives

$$f : A \to B \Rightarrow \alpha(f) = (f, \partial f) : \alpha(A) \to \alpha(B)$$
$$\partial f : A \times \Delta A \to \Delta B \Rightarrow \alpha(\partial f) = (\partial f, \partial^2 f) : \alpha(A \times \Delta A) \to \alpha(\Delta B)$$
$$\partial^2 f : (A \times \Delta A) \times (\Delta A \times \Delta^2 A) \to \Delta^2 B \Rightarrow \dots$$

- "Structure" maps are all differentiable
  - $\partial\oplus, \partial+, \dots$
- "Tangent bundle" functor $\mathbf{T}$ (in fact a monad)

$$\mathbf{T}A = A \times \Delta A$$
$$\mathbf{T}f = \langle f \circ \pi_1, \partial f \rangle$$

Internalization

Whenever **C** is a CCC, there is a morphism $\mathbf{d} : (A \Rightarrow B) \to (A \times \Delta A) \Rightarrow \Delta B$ such that, for every map $f : A \to B$, we have

$$\mathbf{d} \circ \Lambda f = \Lambda(\partial f)$$

Essentially: *the derivative operator is itself a* **C***-map*

Lemma

When **T** is representable, the tangent bundle $\mathbf{T}(A \Rightarrow B)$ is naturally isomorphic to $A \Rightarrow \mathbf{T}B$. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}(A \Rightarrow B) & \xrightarrow{\cong} & A \Rightarrow \mathrm{T}(B) \\ {\scriptstyle \oplus_{A \Rightarrow B}} \downarrow & \swarrow {\scriptstyle \mathbf{Id}_{A} \Rightarrow \oplus_{B}} & \\ A \Rightarrow B & & \end{array}$$

Examples of change action models

Are there actually any such objects? Yes!

- "Free" models
- Cartesian differential categories (somewhat)
- Calculus on groups
- Commutative Kleene algebras

"Free" models

- Problem: $\mathrm{CAct}(\mathbf{C})$ doesn't have enough "higher" structure
- Solution: just add it!

ω-change actions

The category of ω-change actions on $\mathbf{C}$ $\mathrm{CAct}_\omega(\mathbf{C})$ is defined as the limit in $\mathbf{Cat}_\times$ of the following diagram:

$$\begin{array}{ccccccc}
\mathrm{CAct}_\omega(\mathbf{C}) & & & & & & \\
\downarrow & \searrow & & \searrow & & & \\
\mathrm{CAct}(\mathbf{C}) & \underset{\xi}{\overset{\varepsilon}{\leftleftarrows}} & \mathrm{CAct}^2(\mathbf{C}) & \underset{\xi}{\overset{\varepsilon}{\leftleftarrows}} & \mathrm{CAct}^3(\mathbf{C}) & \underset{\xi}{\overset{\varepsilon}{\leftleftarrows}} & \ldots
\end{array}$$

When you unpack it - very similar to Fáa di Bruno (Cockett, Seely 2011)

"Free" models

Two "forgetful" functors

$$\varepsilon : \mathrm{CAct}(\mathbf{C}) \to \mathbf{C}$$
$$\varepsilon(A, \Delta A, \oplus, +, 0) = A$$
$$\xi : \mathrm{CAct}^2(\mathbf{C}) \to \mathrm{CAct}(\mathbf{C})$$
$$\xi((A, \ldots), (\Delta A, \ldots), \oplus, +, 0) = (A, \Delta A, \oplus, +, 0)$$

Intuitively: ε forgets the higher structure, ξ prefers it

The canonical model

There is a "canonical" change action model $\gamma : \mathrm{CAct}_\omega(\mathbf{C}) \to \mathrm{CAct}(\mathrm{CAct}_\omega(\mathbf{C}))$. Furthermore, whenever **C** is a CCC then so is CAct_ω, and the tangent bundle functor **T** is representable.

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or all change action models on **C**, $\alpha : \mathbf{C} \to \mathrm{CAct}(\mathbf{C})$, there is a unique functor $\alpha_\omega : \mathbf{C} \to \mathrm{CAct}_\omega(\mathbf{C})$ making the following diagram commute

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\quad\alpha\quad} & \mathrm{CAct}(\mathbf{C}) \\
{\scriptstyle \exists!\,\alpha_\omega}\downarrow & & \downarrow{\scriptstyle \mathrm{CAct}(\alpha_\omega)} \\
\mathrm{CAct}_\omega(\mathbf{C}) & \xrightarrow{\quad\gamma\quad} & \mathrm{CAct}(\mathrm{CAct}_\omega(\mathbf{C}))
\end{array}
$$

Intuitively: every change action model on **C** can be understood entirely through its embedding into $\mathrm{CAct}_\omega(\mathbf{C})$

Cartesian differential categories (Blute, Cockett, Seely 2009)

- Axiomatise abstract derivatives
- Examples: smooth maps between vector spaces
- Recent generalisation (Cruttwell 2017)

Generalised Cartesian differential category (Cruttwell 2017)

A generalised Cartesian differential category is a Cartesian category $\mathbf{C}$ and:

- For every object A, a commutative monoid $(L(A), +, 0)$
- For every map $f : A \to B$, a map $Df : A \times L(A) \to L(B)$
- Some equations...

Models from Cartesian differential categories

Lemma

In a GCDC, define the tangent bundle functor $\mathbf{T}$ by:

$$\mathbf{T}A = A \times L(A)$$
$$\mathbf{T}f = \langle f \circ \pi_1, Df \rangle$$

$\mathbf{T}$ is a monad in $\mathbf{C}$

Kleisli category of $\mathbf{T}$: "generalised vector fields"

Theorem

Given a GCDC $\mathbf{C}$, the Kleisli category $\mathbf{C}_{\mathbf{T}A}$ can be extended to a change action model

Calculus on groups

The category CGrp

The category **CGrp** is defined by:

- The objects of **CGrp** are groups (in **Set**)
- A morphism $f : (A, +_A, 0_A) \to (B, +_B, 0_B)$ is a (set-theoretic) function $f : A \to B$

Theorem

The category **CGrp** can be extended to a change action model by defining α : **CGrp** $\to \mathrm{CAct}($**CGrp**$)$ as follows:

- $\alpha(A, +_A, 0_A) = (A, A, +_A, +_A, 0_A)$
- $\alpha(f)(a, \delta a) = -f(a) + f(a + \delta a)$

Calculus on groups

Seemingly trivial, but already studied...under two different names!

Boolean differential calculus (Steinbach 2017)

- Calculus on Boolean algebras
- Treat Boolean algebra like a group with **XOR**
- Differential of f: $f(x)$ **XOR** $f(x$ **XOR** $dx)$
- Precisely derivatives in $(\mathbb{B}, \mathbb{B}, \textbf{XOR}, \textbf{XOR}, 0)$

Calculus of finite differences (Jordan 1965)

- Calculus techniques on integers
- Finite difference operator $\Delta f(x) = f(x+1) - f(x)$
- Precisely derivatives in $(\mathbb{Z}, \mathbb{Z}, +, +, 0)$ evaluated "along" 1
- We recover the chain rule!

Commutative Kleene algebras

Derivatives of polynomials on CKAs

Let $\mathbb{K}$ be a commutative Kleene algebra. Given a polynomial $p = p(\overline{x})$ on $\mathbb{K}$, we define its *i-th derivative* $\frac{\partial p}{\partial x_i}(\overline{x}) \in \mathbb{K}[\overline{x}]$:

$$\frac{\partial p^\star}{\partial x_i}(\overline{x}) = p^\star(\overline{x})\,\frac{\partial p}{\partial x_i}(\overline{x})$$

$$\frac{\partial (p+q)}{\partial x_i}(\overline{x}) = \frac{\partial p}{\partial x_i}(\overline{x}) + \frac{\partial q}{\partial x_i}(\overline{x})$$

$$\frac{\partial (p\,q)}{\partial x_i}(\overline{x}) = p(\overline{x})\frac{\partial q}{\partial x_i}(\overline{x}) + q(\overline{x})\frac{\partial p}{\partial x_i}(\overline{x})$$

Taylor's formula (Hopkins, Kozen 1999)

Whenever $p(x) \in \mathbb{K}\,[x]$, we have $p(a+b) = p(a) + b\frac{\partial p}{\partial x}(a+b)$

Commutative Kleene algebras

Finite powers of Kleene algebras

Let $\mathbb{K}$ be a commutative Kleene algebra. We define the category $\mathbb{K}_\times$ as the Cartesian category whose objects are all finite powers of $\mathbb{K}$ and whose arrows are polynomials on $\mathbb{K}$.

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he category $\mathbb{K}_\times$ can be endowed with a change action model $\alpha : \mathbb{K}_\times \to \mathrm{CAct}(\mathbb{K}_\times)$ given by:

- $\alpha(\mathbb{K}) = (\mathbb{K}, \mathbb{K}, +, +, 0)$
- $\alpha(p(x_1, \ldots, x_n)) = \sum_{i=1}^{n} y_i \frac{\partial p}{\partial x_i}(x_1 + y_1, \ldots, x_n + y_n)$

Future work

So far:

- Change actions as models for H-O differentiation
- Well-behaved, pop up everywhere
- Related to GCDC, but different

In the future:

- Interesting 2-categorical story!
- Calculus - incremental System T?
- Do more geometry with this!
- Gradients
  - $f(a) \oplus \delta b = f(a \oplus \nabla(a, \delta b))$
  - Composes, gives rise to category
  - Related to (Van Laarhoven) lenses