Occlusion Operads for Image Segmentation

Third Symposium on Compositional Structures
University of Oxford

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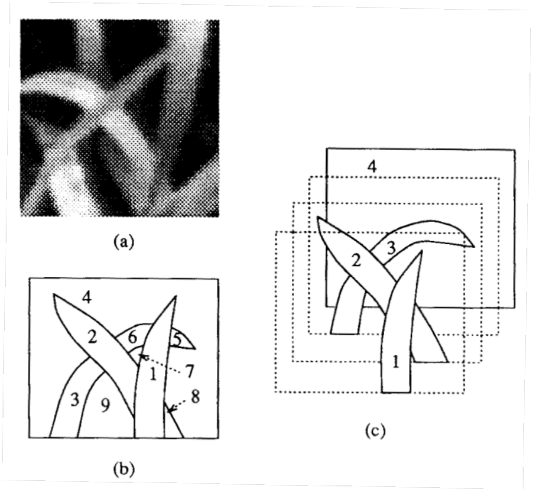
The problem: image segmentation



Reprinted "The 2.1-D Sketch", [Mumford & Nitzberg 1990].

The problem: image segmentation

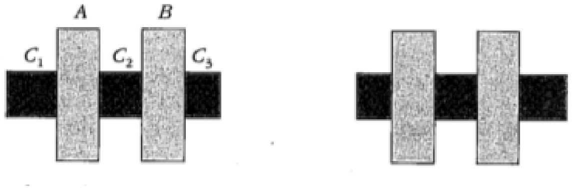


Figure 1: Are C_1, C_2, C_3 distinct objects or part of an occluded one? [Mumford & Nitzberg 1990]

The problem: image segmentation

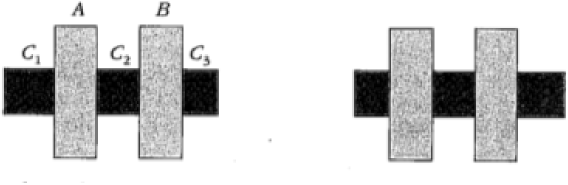


Figure 1: Are C_1, C_2, C_3 distinct objects or part of an occluded one? [Mumford & Nitzberg 1990]

Without the context clues provided by occlusion, the depth of objects relative to others in a scene cannot be determined.

The problem: image segmentation

Forward problem: Given objects in an image, how do they compose to yield patterns of occlusion ordered by depth? What are all possible orderings?

Inverse problem: Given an occlusion pattern for an image, how does it decompose into different occluded segments? What are all possible decompositions?

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Image processing	Monoidal structures
Forward problem	Monoids
Inverse problem	Comonoids
Generating series	Combinatorial species

The problem: image segmentation

Image processing	Monoidal structures
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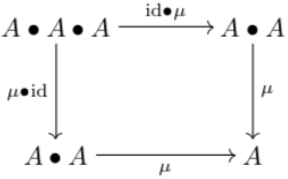
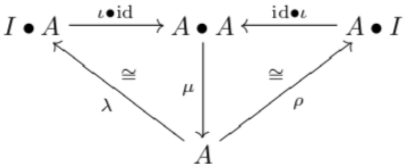
A monoid in a monoidal category $(C, \bullet)$ is a triple (A, μ, ι) such that

$$\mu : A \bullet A \to A \text{ and } \iota : I \to A$$





The problem: image segmentation

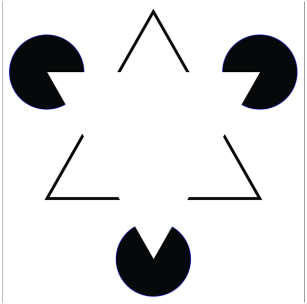


Figure 2: The Kanizsa triangle optical illusion.

The problem: image segmentation

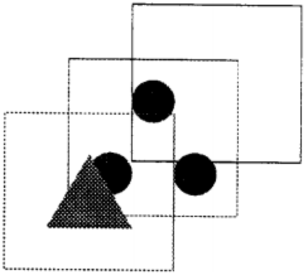


Figure 3: A 2.1-D sketch of the Kanizsa triangle. [Mumford & Nitzberg 1990]

The problem: image segmentation

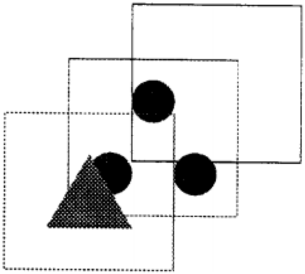


Figure 3: A 2.1-D sketch of the Kanizsa triangle. [Mumford & Nitzberg 1990]

The layers of a 2.1-D sketch comprise ordered partitions, or *set compositions*.

Set compositions and linear orders

Let I be a finite set. A *composition* of I is a finite sequence $(I_1, \ldots, I_k)$ of disjoint nonempty subsets of I such that

$$I = \bigcup_{i=1}^{k} I_i.$$

The subsets I_i are the *blocks* of the composition. We write $F \vDash I$ to indicate that $F = (I_1, \ldots, I_k)$ is a composition of I.

Occlusion monoids

Set monoid example: the "atop monoid" in Haskell's `diagrams`

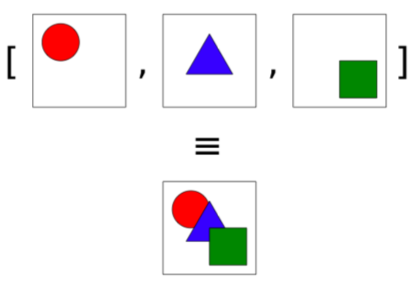


Figure 4: Superimposing a list of primitives, [Yorgey 2012].

Unitality

$$\bigcirc \dashv \varnothing = \varnothing \dashv \bigcirc$$

Associativity

$$\bigcirc \dashv (\triangle \dashv \square) = (\bigcirc \dashv \triangle) \dashv \square$$

Occlusion monoids

A *set species* is a functor

$$\mathbf{q} : \mathsf{Set}^{\times} \to \mathsf{Set}$$

Example: Species of linear orders

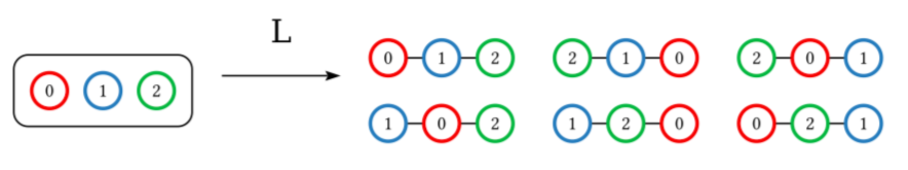


Figure 5: Linear orders on three elements, [Yorgey 2012].

Occlusion monoids

The *linear order species* is a functor from the groupoid of linearly ordered

$$\mathbf{L} \to \mathrm{FinSet}$$

finite sets with order-preserving bijections as morphisms to the category of sets with total functions as morphisms.

The linear order species $\mathbf{L}[I]$ on a finite set of labels I encodes all possible orderings of its elements under a linear order.

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$$\mathbf{L} = 1 + X \cdot \mathbf{L}$$

Recursively,

$$\mathbf{L} = 1 + X + X^2 \ldots$$

Occlusion monoids

Image segmentation	Combinatorial products	Monoidal products
Layer concatenation	Ordinal sum	$\mathbf{L}[S] \dashv \mathbf{L}[T] \to \mathbf{L}[I]$
Segment decomposition	Deshuffling	$\mathbf{L}[I] \to \mathbf{L}_{\vert S} \dashv \mathbf{L}_{\vert T}$

For the species $\mathbf{L}$ of linear orders, we define the product as *concatenation*

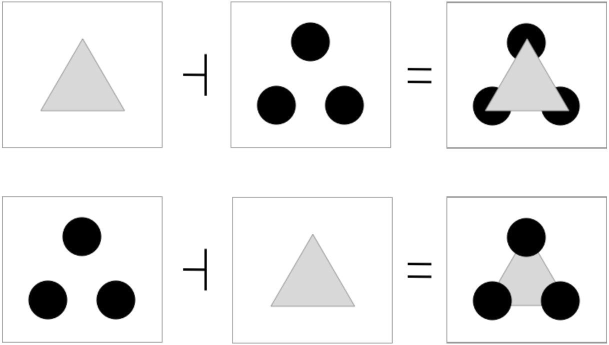
$$\mathbf{L}[S] \otimes \mathbf{L}[T] \to \mathbf{L}[I]$$

$$l_1 \dashv l_2 \mapsto l_1 \cdot l_2$$

and coproduct as *deshuffling*

$$\mathbf{L}[I] \to \mathbf{L}[S] \otimes \mathbf{L}[T]$$

$$l \mapsto l_{\vert S} \dashv l_{\vert T}$$

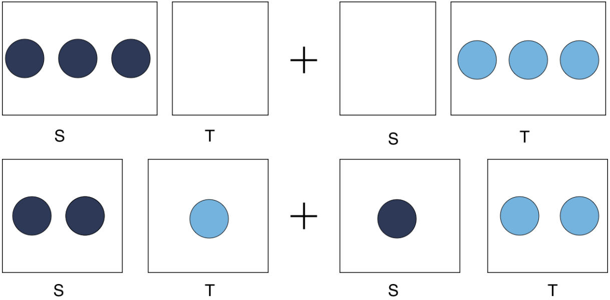


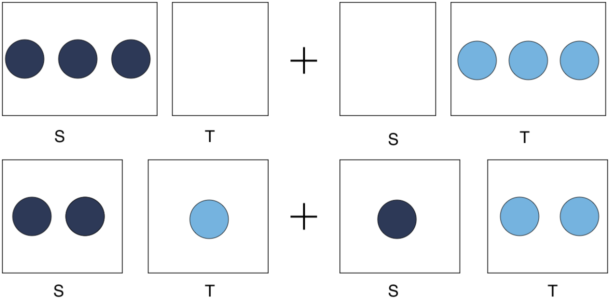
$$\triangle \dashv \bigcirc = l_1 \cdot l_2$$

$$\bigcirc \dashv \triangle = l_2 \cdot l_1$$

Deshuffling via Day convolution

$$\mathbf{F} \cdot \mathbf{G}[I] = \sum_{I = S \sqcup T} F[S] \otimes G[T]$$





Given a set of labels $I = \{a, b, c\}$,

$$\begin{aligned}\mathbf{F} \cdot \mathbf{G}[I] = & (F[abc] \dashv G[\varnothing]) + (F[\varnothing] \dashv G[abc]) \\ & + \ (F[ab] \dashv G[c]) + (F[a] + G[bc]) \\ & + \ (F[c] \dashv G[ab]) + (F[bc] + G[a]) \\ & + \ (F[b] \dashv G[ac]) + (F[ac] + G[b])\end{aligned}$$

Occlusion monoids

The *total preorder species* is a functor from the groupoid of

$$\mathbf{T} \to \mathsf{FinSet}$$

finite sets with totally preordered elements and order-preserving bijections as morphisms to Set.

The total preorder species $\mathbf{T}[I]$ on a finite set of labels I encodes all possible arrangements of its elements under a total preorder.

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Image segmentation	Combinatorial products	Monoidal products
Layer concatenation	Solomon-Tits algebra	$\Sigma[S] \dashv \Sigma[T] \to \Sigma[I]$
Segment decomposition	Deshuffling	$\Sigma[I] \to \Sigma_{\vert S} \dashv \Sigma_{\vert T}$

Generating series

For any species **F**, we have the exponential generating function

$$\mathbf{F}(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

which is a formal power series whose coefficients count **F**-structures.

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$$\mathbf{L}(x) = \sum_{n \geq 0} n! \frac{x^n}{n!} = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

Generating series

We obtain the generating function for **T**

$$\frac{1}{2 - e^x}$$

by substituting the e.g.f. for the uniform nonempty species, $e^x - 1$, into the o.g.f. for the linear order species,

$$\frac{1}{1 - x}$$

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Generating series

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$$\frac{1}{1-x}$$

. We have:

$$\frac{1}{1-(e^x-1)} = \frac{1}{2-e^x}$$

Enumerating occlusions

We enumerate the 13 possible occlusion patterns of the Kanizsa triangle under a total preorder.

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Partitions

$$\sum_{n\geq 1} S(n,k)x^n = e^{e^x-1}, \qquad \text{coeff.} \approx \frac{n!}{(\log n)^n}$$

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Compositions

$$\sum_{n \geq 1} k! S(n,k)x^n = \frac{1}{2 - e^x}, \qquad \text{coeff.} \approx \frac{n!}{2(\log 2)^{n+1}}$$

Consider a total preorder on the set of n elements. The noncommutative operation of occlusion corresponds to strict inequalities, and equality indexes which elements are "tied" or equal in the ordering.

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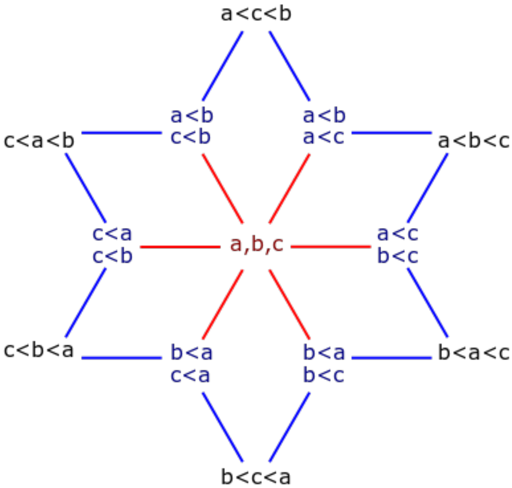

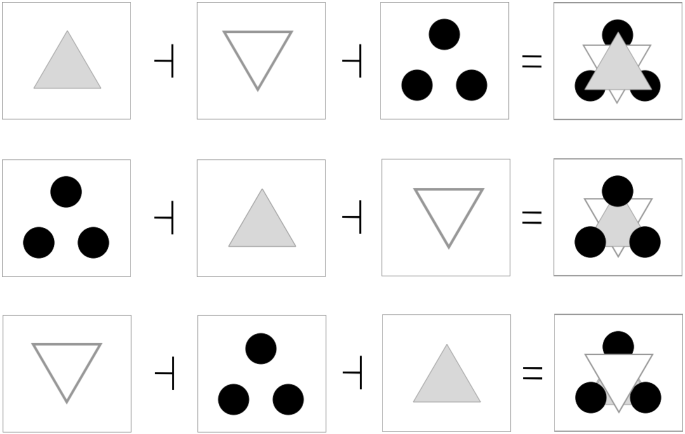

$$\triangle \dashv \nabla \dashv \bigcirc = l_1 \cdot l_2 \cdot l_3$$

$$\bigcirc \dashv \triangle \dashv \nabla = l_3 \cdot l_1 \cdot l_2$$

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Type (3)

$$\triangle \cup \nabla \cup \bigcirc$$

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Then, the 6 linear orders:

Type (1,1,1)

$$\triangle \dashv \nabla \dashv \bigcirc$$

$$\bigcirc \dashv \triangle \dashv \nabla$$

$$\nabla \dashv \bigcirc \dashv \triangle$$

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$$\triangle \dashv \bigcirc \dashv \nabla$$

$$\nabla \dashv \triangle \dashv \bigcirc$$

Finally, the 6 compositions using both occlusion and disjoint union:

Type (2, 1)

$$(\triangle \cup \nabla) \dashv \bigcirc$$

$$(\bigcirc \cup \triangle) \dashv \nabla$$

$$(\nabla \cup \bigcirc) \dashv \triangle$$

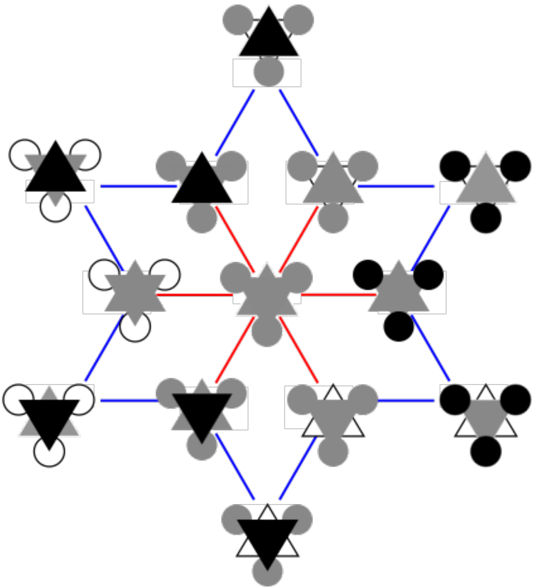
Type (1, 2)

$$\triangle \dashv (\nabla \cup \bigcirc)$$

$$\bigcirc \dashv (\triangle \cup \nabla)$$

$$\nabla \dashv (\bigcirc \cup \triangle)$$

Total preorder on Kanizsa occlusions



Distance between occlusion patterns

Let $I = S \sqcup T$ and $F = (I_1, \ldots, I_k) \vDash I$. The *Schubert statistic* is defined by

$$\mathrm{sch}_{S,T}(F) := |\{(i,j) \in S \times T \mid i \text{ is in a strictly later block of } F \text{ than } j\}|.$$

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Alternatively,

$$\mathrm{sch}_{S,T}(F) = \sum_{1 \leq i < j \leq k} |I_i \cap T|\,|I_j \cap S|.$$

Image segmentation	Combinatorial products	Monoidal products
Pattern difference	Schubert statistic	$\lvert S \times T \rvert,\ \ j < i$ in F

Operads from combinatorial species

Given a composition $F = F^1|\cdots|F^k \vDash I$, we write

$$\mathbf{q}(F) := \mathbf{q}[F^1] \otimes \cdots \otimes \mathbf{q}[F^k]$$

Given a partition X of I, we write

$$\mathbf{q}(X) := \bigotimes_{S \in X} \mathbf{q}[S]$$

These are the unbracketed resp. unordered tensor powers of vector spaces.

Operads from combinatorial species

An operad is a monoid in $(\mathrm{Sp}, \circ, \mathbf{X})$, or a species $\mathbf{p}$ together with morphisms of species $\gamma : \mathbf{p} \circ \mathbf{p} \to p$ and $\eta : \mathbf{X} \to \mathbf{p}$ (which are associative and unital).

This defintion yields a map

$$\gamma_F : \mathbf{p}[X] \otimes \bigotimes_{x \in X} \mathbf{p}[f^{-1}(x)] \to \mathbf{p}[I]$$

Example: Species of linear orders

$$\mathbf{L}[X] \otimes \bigotimes_{S \in X} \mathbf{L}[S] \to \mathbf{L}[I]$$

$$l_x \otimes \bigotimes_{S \in X} l_S \mapsto l_I$$

Thank You!