SYCO 3

Differentiating proofs for programs

Marie Kerjean

Inria Bretagne
1. Linear Logic

2. Smooth classical models

3. LPDEs

4. Higher-Order
A proof is linear when it uses only once its hypothesis $A$. 

$$A \Rightarrow B = !A \rightarrow B$$

$$\mathcal{C}_\infty (A, B) \simeq \mathcal{L}(!A, B)$$
A proof is linear when it uses only once its hypothesis $A$. 

$$A \Rightarrow B = !A \multimap B$$

$$\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$$
Linear logic

Usual implication

Linear implication

Linear Logic

\[ A \Rightarrow B = ! A \multimap B \]
\[ \mathcal{C}^{\infty}(A, B) \simeq \mathcal{L}(!A, B) \]

A proof is linear when it uses only once its hypothesis A.
Linear logic

Linear Logic

\[ A \Rightarrow B = !A \dashv\Leftrightarrow B \]
\[ C^\infty(A, B) \simeq \mathcal{L}(!A, B) \]

A focus on linearity

- Higher-Order is about Seely’s isomorphism.

\[ !A \hat{\otimes}!B \simeq !(A \times B) \]

- Classicality is about a linear involutive negation:

\[ A^\perp\perp \simeq A \]
\[ A \simeq A'' \]
Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$ 

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

  $$\phi_f : g \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$  


- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

  $$!A \multimap \bot = A \Rightarrow \bot$$
Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$ 


- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot \quad \mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$


- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$

$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

$$(!E)^{\prime \prime} \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$ 


- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$

$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

  $$\phi_f : g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \mapsto \int f g.$$ 


- In a classical and Smooth model of Differential Linear Logic, the exponential is a space of distributions with compact support.

  $$!A \multimap \bot = A \Rightarrow \bot$$

  $$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

  $$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

  $$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

- Seely’s isomorphism corresponds to the Kernel theorem:

  $$\mathcal{C}^\infty(E, \mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$
Just a glimpse at Differential Linear Logic

\[ A, B := A \otimes B | 1 | A \& B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A \]

Exponential rules of DiLL₀

\[
\frac{\Gamma \vdash ?A, ?A}{\Gamma \vdash ?A} \quad c
\quad \frac{\Gamma \vdash A}{\Gamma \vdash ?A} \quad w
\quad \frac{\Gamma \vdash A}{\Gamma \vdash ?A} \quad d
\]

\[
\frac{\Gamma \vdash !A, \Delta \vdash !A}{\Gamma, \Delta \vdash !A} \quad \bar{c}
\quad \frac{\Gamma \vdash !A}{\Gamma \vdash !A} \quad \bar{w}
\quad \frac{\Gamma, A \vdash \Delta, !A}{\Gamma, \Delta, !A} \quad \bar{d}
\]

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)
Differential Linear Logic

\[
\frac{\Gamma, A \bot}{\Gamma, \top A \bot} \quad d
\]

\[
\frac{\Gamma, A \bot}{\Gamma, ?A \bot} \quad \overline{d}
\]

A linear proof is in particular non-linear.

\[
\frac{\Delta, A}{\Delta, !A} \quad \overline{d}
\]

From a non-linear proof we can extract a linear proof.

\[ f \in C^\infty(\mathbb{R}, \mathbb{R}) \]

\[ d(f)(0) \]

Differential interaction nets, Ehrhard and Regnier, TCS (2006)
Differential Linear Logic

\[
\frac{\Gamma, \ell : A \bot}{\Gamma, \ell : ? A \bot} \quad d
\]

A linear proof is in particular non-linear.

\[
\frac{\Delta, v : A}{\Delta, (f \mapsto D_0(f)(v)) : !A} \quad d
\]

From a non-linear proof we can extract a linear proof

\[ f \in C^\infty(\mathbb{R}, \mathbb{R}) \]

\[ d(f)(0) \]

Differential interaction nets, Ehrhard and Regnier, TCS (2006)
Differential Linear Logic

\[
\vdash \Gamma, \ell : A_\perp \quad d
\]

\[
\vdash \Gamma, \ell : ?A_\perp \quad d
\]

A linear proof is in particular non-linear.

\[
\vdash \Delta, v : A \\
\vdash \Delta, (f \mapsto D_0(f)(v)) : !A \\
\downarrow d
\]

From a non-linear proof we can extract a linear proof.

Cut-elimination:

\[
\vdash \Gamma, \nu : !A \\
\vdash \Gamma, A_\perp \\
\vdash \Delta, ?A_\perp \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, A \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \\
\vdash \Delta, A_\perp \\
\vdash \Gamma, \Delta \"]

Differential interaction nets, Ehrhard and Regnier, TCS (2006)
Differential Linear Logic

\[ \vdash \Gamma, \ell : A^\perp \]

\[ \vdash \Gamma, \ell : ?A^\perp \]

A linear proof is in particular non-linear.

\[ \vdash \Delta, v : A \]

\[ \vdash \Delta, (f \mapsto D_0(f)(v)) : !A \]

From a non-linear proof we can extract a linear proof.

Cut-elimination:

\[ \vdash \Gamma, \ell : A^\perp \]

\[ \vdash \Delta, \ell : A^\perp \]

\[ \vdash \Delta, (f \mapsto D_0(f)(v)) : !A \]

Differential interaction nets, Ehrhard and Regnier, TCS (2006)
The computational content of differentiation

Historically, resource sensitive syntax and discrete semantics

- Quantitative semantics: \( f = \sum_n f_n \)
- Probabilistic Programming and Taylor formulas: \( M = \sum_n M_n \)

[Ehrhard, Pagani, Tasson, Vaux, Manzonetto ...]

Differentiation in Computer Science can have a different flavour:

- Numerical Analysis and **functional analysis**
- Ordinary and **Partial Differential Equations**
The computational content of differentiation

Historically, resource sensitive syntax and discrete semantics

- Quantitative semantics: \( f = \sum_n f_n \)
- Probabilistic Programming and Taylor formulas: \( M = \sum_n M_n \)
  \[\text{[Ehrhard, Pagani, Tasson, Vaux, Manzonetto ...]}\]

Differentiation in Computer Science can have a different flavour:

- Numerical Analysis and functional analysis
- Ordinary and Partial Differential Equations

Can we match the requirement of models of LL with the intuitions of physics?
(YES, we can.)
Smooth and classical models of Differential Linear Logic

What’s the good category in which we interpret formulas?
Smoothness and Duality

Smoothness

Spaces: $E$ is a **locally convex** and **Haussdorf** topological vector space. 
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

The two requirements works as opposite forces.

✓ A cartesian closed category with smooth functions.  
   $\rightsquigarrow$ **Completeness**, and a dual topology fine enough.

✓ Interpreting $(E^\perp)^\perp \simeq E$ without an orthogonality:  
   $\rightsquigarrow$ **Reflexivity** : $E \simeq E''$, and a dual topology coarse enough.
What’s not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

\[ \dim C^0(\mathbb{R}^n, \mathbb{R}^m) = \infty. \]
What’s not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

\[ \dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty. \]

We can’t restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Girard’s Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can’t be bounded on an unbounded space (Liouville’s Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don’t work.
What’s not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

\[ \dim C^0(\mathbb{R}^n, \mathbb{R}^m) = \infty. \]

We can’t restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Girard’s Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can’t be bounded on an unbounded space (Liouville’s Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don’t work.

We can’t restrict ourselves to normed spaces.
MLL in TopVect

Duality is not an orthogonality in general:

- It depends of the topology $E'_\beta$, $E'_c$, $E'_w$, $E'_\mu$ on the dual.
- It is typically not preserved by $\otimes$.
- It is in the canonical case not an orthogonality: $E'_\beta$ is not reflexive.

Monoidal closedness does not extends easily to the topological case:

- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathcal{L}_B(E \otimes_B F, G) \simeq \mathcal{L}_B(E, \mathcal{L}_B(F, G))$
  $\Leftrightarrow$ ”Grothendieck problème des topologies”.
Topological models of DiLL

[Ehr02] [Ehr05] [DE08]
countable bases
of vector spaces

\( C^\infty(\mathbb{R}^n, \mathbb{R}) \) is not finite dimensional

Reflexive and complete:
e.g. \( C^\infty(\mathbb{R}^n, \mathbb{R}) \)

Coherent Banach spaces [Gir99]
a norm is too restrictive
Fréchet and DF spaces

- Fréchet: metrizable complete spaces.
- (DF)-spaces: such that the dual of a Fréchet is (DF) and the dual of a (DF) is Fréchet.

These spaces are in general not reflexive.
A polarized model of Smooth differential Linear Logic

*Typical Nuclear Fréchet spaces are spaces of [smooth, holomorphic, rapidly decreasing …] functions.*

And more: ↑ is the completion ~ Chiralities [Mellies].
Higher-Order: how do we construct $\mathcal{C}^\infty(\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), \mathbb{R})$.

Partial Differentiation Equations: Distribution theory allows to generalize the interaction between linearity and non-linearity to the interaction between the solutions and the parameters to a differential equation.

$\leadsto$ interactions between theoretical computer science and applied mathematics.
A Logical account for Linear Partial Differential Equations
Linear functions as solutions to a Differential equation

**Slogan**: From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

\[ f \in C^\infty(A, \mathbb{R}) \text{ is linear} \iff \forall x, f(x) = D_0(f)(x) \]
\[ \iff \exists g \in C^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d}g \]
\[ \phi \in A'' \simeq A \iff \exists \psi \in !A, D_0(\phi) = \psi \]
\[ \phi \in !D A \iff \exists \psi \in !A, D(\phi) = \psi \]
Linear functions as solutions to a Differential equation

**Slogan**: From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

\[ f \in C^\infty(A, \mathbb{R}) \text{ is linear } \iff \forall x, f(x) = D_0(f)(x) \]

\[ \phi \in A'' \simeq A \iff \exists \psi \in !A, D_0(\phi) = \psi \]

\[ \phi \in !_D A \iff \exists \psi \in !A, D(\phi) = \psi \]

\[ \bar{d}: \begin{cases} E'' \to C^\infty(E, \mathbb{R})', \\ \phi = ev_x \mapsto \phi \circ D_0 = (f \mapsto ev_x(D_0(f)) \end{cases} \]

\[ d: \begin{cases} !E \to E'' \\ \psi \mapsto \psi|_{E'} \end{cases} \]

As \( \mathcal{L}(E, \mathbb{R}) = D_0(C^\infty(E, \mathbb{R})) \):

\[ \bar{d}: \begin{cases} (D_0(C^\infty(E, \mathbb{R})))' \to C^\infty(E, \mathbb{R})', \\ \phi \mapsto \phi \circ D_0 \end{cases} \]

\[ d: \begin{cases} C^\infty(E, \mathbb{R})' \to (D_0(C^\infty(E, \mathbb{R})))' \\ \psi \mapsto \psi|_{D_0(C^\infty(E, \mathbb{R}))} \]
Dereliction and co-dereliction, again.

\[ \bar{d} : \begin{cases} (D_0(C^\infty(E, \mathbb{R}))')' \to C^\infty(E, \mathbb{R})', \\
\phi \mapsto \phi \circ D_0 \end{cases} \]

\[ d : \begin{cases} C^\infty(E, \mathbb{R})' \to (D_0(C^\infty(E, \mathbb{R}))')' \\
\psi \mapsto \psi|_{D_0(C^\infty(E, \mathbb{R}))} \end{cases} \]

\[ \bar{d}_D : \begin{cases} (D(C^\infty(E, \mathbb{R}))')' \to C^\infty(E, \mathbb{R})' \\
\phi \mapsto \phi \circ D \end{cases} \]

\[ d_D : \begin{cases} C^\infty(E, \mathbb{R})' \to (D(C^\infty(E, \mathbb{R}))')' \\
\psi \mapsto \psi|_{D(C^\infty(E, \mathbb{R}))} \end{cases} \]

Another exponential is possible

\[ !D E := D^{-1}((C^\infty(E, \mathbb{R}))') \subset (C^\infty_c(E, \mathbb{R}))' \]

The exponential is the space of solutions to a differential equation.

- \[ !D_0 E := E'' \simeq E. \]
- \[ !Id E := !E = C^\infty(E, \mathbb{R})'. \]
Linear Partial Differential Equations with constant coefficient

Consider $D$ a LPDO with constant coefficients:

$$D = \sum_{\alpha, |\alpha| \leq n} a_{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}}.$$

The heat equation in $\mathbb{R}^2$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

$$u(x, y, 0) = f(x, y)$$

**Theorem (Malgrange 1956)**

For any $D$ LPDOcc, there is $E_D \in C_{c}^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})'$ such that:

$$D(E_D) = \delta_0$$

and thus: output $D(E_D \ast \phi) = \phi$ input
A deterministic cut-elimination.

A Logical Account for LPDEs, K. LICS 2018.
How to compute with higher-order distributions?
joint work with JS Lemay.
Finite dimensional vector spaces into $E$

For every linear continuous injective function $f : \mathbb{R}^n \to E$:

$$E'_f(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n)'$$

Higher-order distributions

$$E'(E) := \lim_{\longrightarrow} E'_f(\mathbb{R}^n)$$

directed under the inclusion maps defined as

$$S_{f,g} : E'_g(\mathbb{R}^n) \to E'_f(\mathbb{R}^m), \phi \mapsto (h \mapsto \phi(h \circ \iota_{n,m}))$$

when $f = g \circ \iota_{n,m}$.

functorial only on injective linear maps: no promotion.
All about reflexivity

When $E$ is reflexive, so is $\mathcal{E}'(E)$.

Duality works well:

$$\mathcal{E}'(E) \simeq \left( \lim_{f: \mathbb{R}^n \to E} \mathcal{E}_f(\mathbb{R}^n) \right)'$$

but we still are in a polarized model.

A strong monoidal functor on isomorphisms

\[
! : \begin{cases} 
\text{REFL}_{iso} \to \text{REFL}_{iso} \\
E \mapsto \mathcal{E}'(E) \\
\ell : E \to F \mapsto !\ell \in \mathcal{E}(F')
\end{cases}
\]

where $!\ell(f_f) = f_{\ell \circ f : \mathbb{R}^n \to F}$. 
Higher-order dereliction and co-dereliction

\[ d_E : \begin{cases} 
!(E) \to E'' \simeq E \\
\phi \mapsto (\ell \in E' \mapsto \phi((\ell \circ f)_{f: \mathbb{R}^n \to E} \in \mathcal{E}(E)) 
\end{cases} \]

\[ \overline{d}_E : \begin{cases} 
E \to !E \simeq (\mathcal{E}(E))' \\
x \mapsto (f_f \in C_f^\infty(\mathbb{R}^n, \mathbb{R}))_{f: \mathbb{R}^n \to E'} \mapsto D_0 f_f(f^{-1}(x)) 
\end{cases} \]

where \( f \) is injective such that \( x \in \text{Im}(f) \).
Computing in higher-dimension

\[ !E = \langle \delta_x, x \in E \rangle \]

By Frölicher, as used by Blute, Ehrhard and Tasson.

That’s a *discretisation scheme* :

let’s embed numerical schemes into cut-elimination, through compositionality.
Conclusion

Polarization

*From mathematics to proof-theory and back.*

Differential Equations
A coalgebraic structure on $D$

**Weakening**

$w : !_DE \to \mathbb{R}$ comes from $t : E \to \{0\}$.

If $E = \mathbb{R}^n$, define $\mathbb{R}^{n'}$ another copy of $E$. Then

\[
D(C^\infty(E, \mathbb{R})) \to D(C^\infty(E \times E, \mathbb{R})) \\
= D(C^\infty(\mathbb{R}^n \times \mathbb{R}^{n'}, \mathbb{R})) \\
= D(C^\infty(E, \mathbb{R}) \otimes C^\infty(\mathbb{R}^{n'}, \mathbb{R})) \\
= D(C^\infty(E, \mathbb{R})) \otimes C^\infty(\mathbb{R}^{n'}, \mathbb{R})
\]

**Contraction**

We thus have $c : !_DE \to !_E \otimes !_DE$. 
What’s typable with D-DiLL

Consider $D$ a Smooth Linear Partial Differential Operator: $D : \mathcal{C}^\infty(E) \rightarrow \mathcal{C}^\infty(E)$. $D$ acts on $E \times E$:

$$\hat{D} = (D \otimes \text{Id}_F)\mathcal{C}^\infty(E \times E, \mathbb{R}) \rightarrow \mathcal{C}^\infty(E \times E, \mathbb{R})$$

Then Green’s function is the operator $K_{x,y} : !E \rightarrow E$ such that:

$$K_{x,y} \circ (\hat{D})' = \delta_{x-y}$$

$\frac{\vdash \Gamma, ?_D E^\perp, ? E^-}{\vdash ?_D E^\perp} c_D$ \hspace{1cm} $\frac{\vdash \Delta, ?_D E}{\vdash !_D E} c_D$

$\frac{\vdash \Delta, ?_D E}{\vdash ?_D \Delta, !_D E}$ cut

$\frac{\vdash \Gamma, \Delta}{\vdash \Gamma, \Delta}$ cut
A closer look to Kernels

A answer to a well-known issue :

- Any \( k \in (L_p(\mu \otimes \eta))' \) gives rise to a compact operator
  \( T_k : L_p(\mu) \to L_{p^*}(\eta) \cong (L_p(\eta))' : T_k(f)(g) = k(f.g) \).
- This is not a surjection : if \( p = p^* = 2 \), for \( T_k = \text{Id} \) one should have
  \( k = \delta_{x-y} \), which is not a function.
- The above morphism \( k \mapsto T_k \) is an isomorphism on spaces of distributions
  spaces, generalizing \( L_p \) :

Kernel theorems

\[
\mathcal{L}(C^\infty(E, \mathbb{R})', C^\infty(F, \mathbb{R})'') \cong C^\infty(E, \mathbb{R})' \hat{\otimes} C^\infty(F, \mathbb{R})'
\cong C^\infty(E \times F, \mathbb{R})'

T_k \mapsto K_{x,y}
\]
A closer look to Kernels

A answer to a well-known issue:

- Any $k \in (L_p(\mu \otimes \eta))'$ gives rise to a compact operator $T_k : L_p(\mu) \to L_p^*(\eta) \simeq (L_p(\eta))' : T_k(f)(g) = k(f.g)$.

- This is not a surjection: if $p = p^* = 2$, for $T_k = Id$ one should have $k = \delta_{x-y}$, which is not a function.

- The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions spaces, generalizing $L_p$:

Kernel theorems

$$C^\infty(E, \mathbb{R})' \hat{\otimes} C^\infty(F, \mathbb{R})' \simeq \mathcal{L}(C^\infty(E, \mathbb{R})', C^\infty(F, \mathbb{R})'') \simeq C^\infty(E \times F, \mathbb{R})'$$

Nuclearity
A closer look to Kernels

A answer to a well-known issue:

▶ Any $k \in (L_p(\mu \otimes \eta))'$ gives rise to a compact operator

$T_k : L_p(\mu) \rightarrow L_{p^*}(\eta) \simeq (L_p(\eta))' : T_k(f)(g) = k(f.g)$.

▶ This is not a surjection: if $p = p^* = 2$, for $T_k = Id$ one should have

$k = \delta_{x-y}$, which is not a function.

▶ The above morphism $k \mapsto T_k$ is an isomorphism on spaces of distributions

spaces, generalizing $L_p$:

Kernel theorems

$$C^\infty(E, \mathbb{R})' \hat{\otimes} C^\infty(F, \mathbb{R})' \simeq \mathcal{L}(C^\infty(E, \mathbb{R})', C^\infty(F, \mathbb{R})'')$$

$$\simeq C^\infty(E \times F, \mathbb{R})'$$

Density