Probabilistic Open Games

Neil Ghani, Clemens Kupke, Alasdair Lambert, Fredrik Nordvall Forsberg

University Of Strathclyde

SYCO3, Oxford, 28 March 2019

Game Theory

What is Game Theory?

- The mathematical study of strategic interaction between rational agents

What is Game Theory?

- The mathematical study of strategic interaction between rational agents
- Agents pick a strategy to play

What is Game Theory?

- The mathematical study of strategic interaction between rational agents
- Agents pick a strategy to play
- The outcome is determined by collective action of all agents

What is Game Theory?

- The mathematical study of strategic interaction between rational agents
- Agents pick a strategy to play
- The outcome is determined by collective action of all agents
- The outcome determines the utility each agent receives

What is Game Theory?

- The mathematical study of strategic interaction between rational agents
- Agents pick a strategy to play
- The outcome is determined by collective action of all agents
- The outcome determines the utility each agent receives
- Analyse these games via equilibrium

What is an equilibrium?

(σ_0, σ_1) Nash Equilibrium if

- $\sigma_0 \in \arg\max_{\sigma' \in \Sigma_0} \{u_0(\sigma', \sigma_1)\}$; and
- $\sigma_1 \in \arg\max_{\sigma'' \in \Sigma_1} \{u_1(\sigma_0, \sigma'')\}$

Prisoner's dilemma

		Player 2: C	Player 2: D
Player 1	C	$3, 3$	$0, 4$
Player 1	D	$4, 0$	$1, 1$

Prisoner's dilemma

Player 2

		C	D
Player 1	C	$3, 3$	$0, 4$
	D	$4, 0$	$1, 1$

Only equilibrium: (D, D).

Matching Pennies

		Bob	
		H	T
Alice	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

Matching Pennies

		Bob	
		H	T
Alice	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

No equilibrium.

Matching Pennies

		Bob	
		H	T
Alice	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

No pure equilibrium.

Matching Pennies

		Bob	
		H	T
Alice	H	$-1, 1$	$1, -1$
	T	$1, -1$	$-1, 1$

No pure equilibrium.

Only mixed equilibrium: both play $\frac{1}{2}H + \frac{1}{2}T$.

Problems with Game Theory

- Complexity issues
- Finding equilibria is computationally hard
- Games do not compose

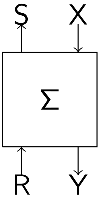
Pure Open Games

Pure Open Games [Hedges 2016]

Neil Ghani, Jules Hedges, Viktor Winschel, Philipp Zahn
Compositional game theory.
LICS 2018.

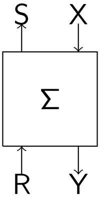
- A framework for building games compositionally
- Applying Category Theory to Game Theory

Pure open games: definition



Let X, Y, R and S be sets. A *pure open game* $G : (X, S) \to (Y, R)$ consists of:

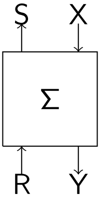
Pure open games: definition



Let X, Y, R and S be sets. A *pure open game* $G : (X, S) \to (Y, R)$ consists of:

- a set Σ of *strategy profiles* for G

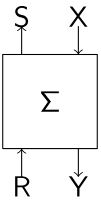
Pure open games: definition



Let X, Y, R and S be sets. A *pure open game* $G : (X, S) \to (Y, R)$ consists of:

- a set Σ of *strategy profiles* for G
- a *play function* $P : \Sigma \times X \to Y$

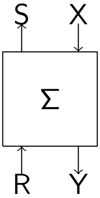
Pure open games: definition



Let X, Y, R and S be sets. A *pure open game* $G : (X, S) \to (Y, R)$ consists of:

- a set Σ of *strategy profiles* for G
- a *play function* $P : \Sigma \times X \to Y$
- a *coutility function* $C : \Sigma \times X \times R \to S$

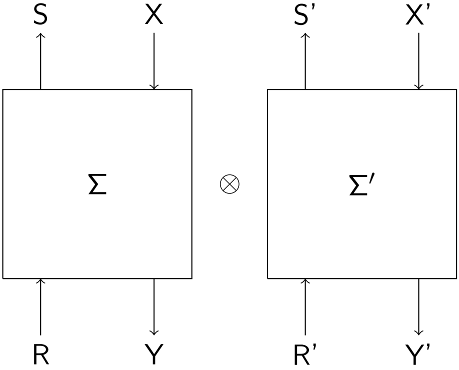
Pure open games: definition



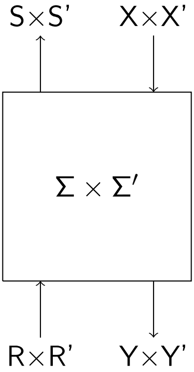
Let X, Y, R and S be sets. A *pure open game* $G : (X, S) \to (Y, R)$ consists of:

- a set Σ of *strategy profiles* for G
- a *play function* $P : \Sigma \times X \to Y$
- a *coutility function* $C : \Sigma \times X \times R \to S$
- an *equilibrium function* $E : X \times (Y \to R) \to \mathscr{P}(\Sigma)$.

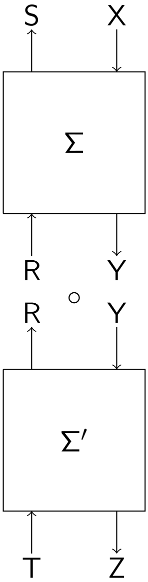
Pure open games: parallel composition



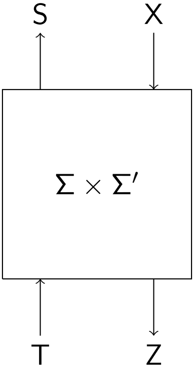
Pure open games: parallel composition



Pure open games: sequential composition



Pure open games: sequential composition



Incorporating mixed strategies

- Want to also capture mixed strategies.
- Solution: use the distributions monad for categorical probability theory [Perrone 2018].

Commutative Monads

Monads and strength

- A *strong monad* on a monoidal category $\mathbb{C}$ is a monad (T, η, μ) with a left strength $s_l : A \otimes TB \to T(A \otimes B)$.

Monads and strength

- A *strong monad* on a monoidal category $\mathbb{C}$ is a monad (T, η, μ) with a left strength $s_l : A \otimes TB \to T(A \otimes B)$.

- If $\mathbb{C}$ is symmetric monoidal, we can define a right strength $s_r : TA \otimes B \to T(A \otimes B)$ by

$$TA \otimes B \xrightarrow{\gamma} B \otimes TA \xrightarrow{s_l} T(B \otimes A) \xrightarrow{T\gamma} T(A \otimes B)$$

Commutative monads

A strong monad on a symmetric monoidal category is *commutative* if

$$\begin{array}{ccccc} TA \otimes TB & \xrightarrow{s_l} & T(TA \otimes B) & \xrightarrow{Ts_r} & TT(A \otimes B) \\ \downarrow{\scriptstyle s_r} & & & & \downarrow{\scriptstyle \mu} \\ T(A \otimes TB) & \xrightarrow{Ts_l} & TT(A \otimes B) & \xrightarrow{\mu} & T(A \otimes B) \end{array}$$

We call this map $\ell : TA \otimes TB \to T(A \otimes B)$.

The finite distribution monad $\mathscr{D} : \mathsf{Set} \to \mathsf{Set}$

Probability distribution on X:

- function $\omega : X \to [0, 1]$
- $\sum_x \omega(x) = 1$
- finite support.

The finite distribution monad $\mathscr{D} : \text{Set} \to \text{Set}$

Probability distribution on X:

- function $\omega : X \to [0, 1]$
- $\sum_x \omega(x) = 1$
- finite support.

$\mathscr{D}(X)$ collection of distributions on X.

- $\eta : X \to \mathscr{D}X$ point distribution.
- $\mu : \mathscr{D}^2 X \to \mathscr{D}X$ flattens distributions of distributions.
- $\ell : \mathscr{D}X \times \mathscr{D}Y \to \mathscr{D}(X \times Y)$ independent joint distribution.
- $\mathscr{D}$-algebras: convex sets R, with "expectation" $\mathbb{E} : \mathscr{D}R \to R$.

Probabilistic Open Games

Probabilistic Open Games

Let X and Y be sets, and R and S $\mathscr{D}$-algebras. A probabilistic open game $G : (X, S) \to (Y, R)$ consists of

Probabilistic Open Games

Let X and Y be sets, and R and S $\mathscr{D}$-algebras. A probabilistic open game $G : (X, S) \to (Y, R)$ consists of

- a set Σ of strategies

Probabilistic Open Games

Let X and Y be sets, and R and S $\mathscr{D}$-algebras. A probabilistic open game $G : (X, S) \to (Y, R)$ consists of

- a set Σ of strategies
- a play function $P : \Sigma \times X \to Y$

Probabilistic Open Games

Let X and Y be sets, and R and S $\mathscr{D}$-algebras. A probabilistic open game $G : (X, S) \to (Y, R)$ consists of

- a set Σ of strategies
- a play function $P : \Sigma \times X \to Y$
- a coutility function $C : \Sigma \times X \times R \to S$

Probabilistic Open Games

Let X and Y be sets, and R and S $\mathscr{D}$-algebras. A probabilistic open game $G : (X, S) \to (Y, R)$ consists of

- a set Σ of strategies
- a play function $P : \Sigma \times X \to Y$
- a coutility function $C : \Sigma \times X \times R \to S$
- an equilibrium function $E : X \times (Y \to R) \to \mathscr{P}(\mathscr{D}\Sigma)$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\Phi \in E_{G \otimes H}\ (x_1, x_2)\ k \quad \text{iff}$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\Phi \in E_{G \otimes H}\ (x_1, x_2)\ k \ \text{ iff } \Phi = \ell(\phi_1, \phi_2) \text{ and}$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\Phi \in E_{G \otimes H}\ (x_1, x_2)\ k \ \text{ iff } \Phi = \ell(\phi_1, \phi_2) \text{ and}$$
$$\phi_1 \in E_G$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\Phi \in E_{G \otimes H}\ (x_1, x_2)\ k \ \text{ iff } \Phi = \ell(\phi_1, \phi_2) \text{ and}$$
$$\phi_1 \in E_G\ x_1$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G \otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\Phi \in E_{G \otimes H}\ (x_1, x_2)\ k \ \text{ iff } \Phi = \ell(\phi_1, \phi_2) \text{ and}$$
$$\phi_1 \in E_G\, x_1\ \mathbb{E}[\mathscr{D}(\pi_0) \circ \mathscr{D}(k) \circ \ell(\eta\_, \mathscr{D}(P_H(\_, x_2))\phi_2)] \text{ and}$$

Parallel composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (X', S') \to (Y', R')$ we need to define the equilibrium

$$E_{G\otimes H} : X \times X' \times (Y \times Y' \to R \times R') \to \mathscr{P}(\mathscr{D}(\Sigma \times \Sigma'))$$

$$\begin{aligned}
\Phi \in & E_{G\otimes H}\ (x_1, x_2)\ k \ \text{ iff } \Phi = \ell(\phi_1, \phi_2) \text{ and} \\
\phi_1 \in & E_G\, x_1\, \mathbb{E}[\mathscr{D}(\pi_0) \circ \mathscr{D}(k) \circ \ell(\eta_, \mathscr{D}(P_H(_, x_2))\phi_2)] \text{ and} \\
\phi_2 \in & E_H\, x_2\, \mathbb{E}[\mathscr{D}(\pi_1) \circ \mathscr{D}(k) \circ \ell(\mathscr{D}(P_G(_, x_1))\phi_1, \eta_)]
\end{aligned}$$

Independent strategies

- Φ is an independent joint distribution $\Phi = \ell(\phi_1, \phi_2)$:

$$\Phi(\sigma, \sigma') = \phi_1(\sigma)\phi_2(\sigma')$$

Independent strategies

- Φ is an independent joint distribution $\Phi = \ell(\phi_1, \phi_2)$:

$$\Phi(\sigma, \sigma') = \phi_1(\sigma)\phi_2(\sigma')$$

- No collusion between players.

Independent strategies

- Φ is an independent joint distribution $\Phi = \ell(\phi_1, \phi_2)$:

$$\Phi(\sigma, \sigma') = \phi_1(\sigma)\phi_2(\sigma')$$

- No collusion between players.
- Mathematically: needed for associativity of composition.

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and

$\phi_1 \in E_G$

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and

$\phi_1 \in E_G\, x$

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and

$\phi_1 \in E_G\, x\ \mathbb{E}[\mathscr{D}(C_H)\ell(\phi_2, \ell(\eta\_, \mathscr{D}(k)(P_H \circ \ell(\phi_2, \eta\_))))])$ and

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and
$\phi_1 \in E_G\, x\ \mathbb{E}[\mathscr{D}(C_H)\ell(\phi_2, \ell(\eta\_, \mathscr{D}(k)(P_H \circ \ell(\phi_2, \eta\_))))])$ and
$\phi_2 \in E_H\ ?\ k$

Sequential composition

Play, coplay same as in pure case.

For games $G : (X, S) \to (Y, R)$ and $H : (Y, R) \to (Z, T)$ we need to define the equilibrium $E_{H \circ G} : X \times (Z \to T) \to \mathscr{P}(\Sigma_G \times \Sigma_H)$

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and
$\phi_1 \in E_G\, x\ \mathbb{E}[\mathscr{D}(C_H)\ell(\phi_2, \ell(\eta\_, \mathscr{D}(k)(P_H \circ \ell(\phi_2, \eta\_))))])$ and
$\phi_2 \in E_H\ ?\ k$

how do we produce a state for the second game?

Condition for second game

$$E_H : Y \times (Z \to T) \to \mathscr{P}(\mathscr{D}\Sigma_H)$$

Condition for second game

$$E_H(\_, k) : \quad Y \to \mathscr{P}(\mathscr{D}\Sigma_H)$$

Condition for second game

$$
\begin{aligned}
E_H(\_, k): &\quad Y \to \mathscr{P}(\mathscr{D}\Sigma_H) \\
\mathscr{D}(P_1(\_, x))\phi_1 : \mathscr{D}Y
\end{aligned}
$$

Condition for second game

$$E_H(\_, k): \quad Y \to \mathscr{P}(\mathscr{D}\Sigma_H)$$
$$\mathscr{D}(P_1(\_, x))\phi_1 : \mathscr{D}Y$$

Want to "lift" $E_H(\_, k)$ from inputs in Y to inputs in $\mathscr{D}Y$.

Kleisli relational lifting

$$\frac{R : X \to \mathscr{P}(\mathscr{D}Y)}{\overline{\mathscr{D}}^{\#}(R) : \mathscr{D}X \to \mathscr{P}(\mathscr{D}Y)}$$

Compare:

Kleisli lifting

$$\frac{f : X \to \mathscr{D}Y}{f^{\#} : \mathscr{D}X \to \mathscr{D}Y}$$

Relational lifting

$$\frac{R \in \mathscr{P}(X \times Y)}{\overline{\mathscr{D}}(R) \in \mathscr{P}(\mathscr{D}X \times \mathscr{D}Y)}$$

Constructing $\overrightarrow{\mathscr{D}}^{\#}(R)$

$$\frac{R : X \to \mathscr{P}(\mathscr{D}Y)}{\overrightarrow{\mathscr{D}}^{\#}(R) : \mathscr{D}X \to \mathscr{P}(\mathscr{D}Y)}$$

$$\mathscr{D}X \xrightarrow{\quad \mathscr{D}R \quad} \mathscr{D}\mathscr{P}\mathscr{D}Y \xrightarrow{\lambda} \mathscr{P}\mathscr{D}^2 Y \xrightarrow{\mathscr{P}\mu} \mathscr{P}(\mathscr{D}Y)$$

Constructing $\overrightarrow{\mathscr{D}}^{\#}(R)$

$$\frac{R : X \to \mathscr{P}(\mathscr{D}Y)}{\overrightarrow{\mathscr{D}}^{\#}(R) : \mathscr{D}X \to \mathscr{P}(\mathscr{D}Y)}$$

$$\mathscr{D}X \xrightarrow{\quad \mathscr{D}R \quad} \mathscr{D}\mathscr{P}\mathscr{D}Y \xrightarrow{\lambda} \mathscr{P}\mathscr{D}^2Y \xrightarrow{\mathscr{P}\mu} \mathscr{P}(\mathscr{D}Y)$$

Here

$$\lambda : \mathscr{D}\mathscr{P} \to \mathscr{P}\mathscr{D}$$

distributive law of functors (not of monads! [Zwart and Marsden 2018]).

Sequential composition, take 2

$\Phi \in E_{H \circ G}\, x\ k$ iff $\Phi = \ell(\phi_1, \phi_2)$ and
$\phi_1 \in E_G\ x\ \mathbb{E}[\mathscr{D}(C_H)\ell(\phi_2, \ell(\eta\_, \mathscr{D}(k)(P_H \circ \ell(\phi_2, \eta\_))))])$ and

Sequential composition, take 2

$$\Phi \in E_{H \circ G}\, x\ \ k \quad \text{iff } \Phi = \ell(\phi_1, \phi_2) \text{ and}$$
$$\phi_1 \in E_G\ x\ \ \mathbb{E}[\mathscr{D}(C_H)\ell(\phi_2, \ell(\eta\_, \mathscr{D}(k)(P_H \circ \ell(\phi_2, \eta\_))))]) \text{ and}$$
$$\phi_2 \in \overline{\mathscr{D}}^{\#}(E^k)\ \ (\mathscr{D}(P_1(\_, x))\phi_1)$$

Compositional game theory with mixed strategies

Theorem

*Probabilistic open games are the morphisms of a monoidal category, with $\otimes$ and $\circ$ given by parallel and sequential composition.**

* Some details still to be checked.

Matching pennies compositionally

Two (identical) component games $P_1, P_2 : (1, \mathbb{R}) \to (\{H, T\}, \mathbb{R})$ with

Matching pennies compositionally

Two (identical) component games $P_1, P_2 : (1, \mathbb{R}) \to (\{H, T\}, \mathbb{R})$ with

- Strategies $\Sigma = \{H, T\}$

Matching pennies compositionally

Two (identical) component games $P_1, P_2 : (1, \mathbb{R}) \to (\{H, T\}, \mathbb{R})$ with

- Strategies $\Sigma = \{H, T\}$
- play and coutility functions trivial

Matching pennies compositionally

Two (identical) component games $P_1, P_2 : (1, \mathbb{R}) \to (\{H, T\}, \mathbb{R})$ with

- Strategies $\Sigma = \{H, T\}$
- play and coutility functions trivial
- equilibrium maximising expected utility

$$\phi \in E(u) \text{ iff } \phi \in \arg\max_{\phi' \in \mathscr{D}\Sigma} \{\mathbb{E}[\mathscr{D}(u)\phi']\}$$

Matching pennies compositionally

Two (identical) component games $P_1, P_2 : (1, \mathbb{R}) \to (\{H, T\}, \mathbb{R})$ with

- Strategies $\Sigma = \{H, T\}$
- play and coutility functions trivial
- equilibrium maximising expected utility

$$\phi \in E(u) \text{ iff } \phi \in \arg\max_{\phi' \in \mathscr{D}\Sigma} \{\mathbb{E}[\mathscr{D}(u)\phi']\}$$

Theorem

$$MP = P_1 \otimes P_2.$$

Conclusions

- Open games with mixed strategies.
- Parallel and sequential composition.

In the future:

- Infinite games
- Universal properties and adjunctions via 2-cells
- Other commutative monads (quitting games)
- Monad transformers and other solution concepts

Conclusions

- Open games with mixed strategies.
- Parallel and sequential composition.

In the future:

- Infinite games
- Universal properties and adjunctions via 2-cells
- Other commutative monads (quitting games)
- Monad transformers and other solution concepts

Thank you!