From parametricity to modularity and back in correspondence theory: preliminary considerations

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The phenomenon of correspondence

\[ F, w \models □ □ p \rightarrow □ p \quad \text{iff} \quad F \models \forall y, z (xRy \& yRz \rightarrow xRz)[w] \]
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\[ F, w \models \Diamond \Diamond p \rightarrow \Diamond p \quad \text{iff} \quad F \models \forall y, z(xRy \& yRz \rightarrow xRz)[w] \]

(⇒) Assume \( wRy \) and \( yRz \). To show: \( w \in R^{-1}[z] \).
Consider the minimal valuation making the antecedent true at \( w \):

\[ V^*(p) = \{z\}. \]

If \( wRy \) and \( yRz \) then \( F, V^*, w \models \Diamond \Diamond p \). Hence, \( F, V^*, w \models \Diamond \Diamond p \), i.e.

\[ w \in \llbracket \Diamond p \rrbracket_{V^*} = R^{-1}[V^*(p)] = R^{-1}[z]. \]
Correspondence theory

- gives syntactic conditions for modal formulas to have a first order correspondent (e.g. Sahlqvist formulas)
- Computes algorithmically the first order correspondent of these formulas
- Benefits: These formulas generate logics that are strongly complete w.r.t. first-order definable classes of frames.
Correspondence theory arises semantically:

Correspondence via Duality

Kripke Frames

Modal logic

Correspondence

First order logic

Correspondence available not just for modal logic: Propositional logic, Algebra Spaces, AAL Model theory.
Correspondence via Duality

An asymmetry:

- Modal logic
- Non canonical interpretation
- First order logic
- Canonical interpretation

Kripke Frames

Symmetry re-established via duality:

BAOs

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Symmetry re-established via duality:

- BAOs
- Kripke Frames

Modal logic
- First order logic

Specific correspondences as logical reflections of dualities
Dual characterizations as instances of unified correspondence
Correspondence via Duality

Correspondence available not just for modal logic:

- Algebra
- Spaces

AAL
Propositional logic
Correspondence
First order logic
Model theory

Correspondence theory arises semantically:
Kripke Frames
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An asymmetry:
Non canonical interpretation
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Correspondence available not just for modal logic:
Propositional logic
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Correspondence available not just for modal logic:

- specific correspondences as logical reflections of dualities
- dual characterizations as instances of unified correspondence

Diagram:

- Algebra to Spaces via Correspondence
- Spaces to Algebra via Correspondence
- Propositional logic to Model theory via AAL
- First order logic to Algebra via Correspondence

Abbreviations:
- AAL
- BAOs
Unified correspondence

Display calculi [GMPTZ18]
Normal (D)LE-logics [CR17]

Hybrid logics
[CP12, CP19]
Mu-calculi
[CFPS15, CGP14, CC17]

Regular DLE-logics

Kripke frames with impossible worlds
[PSZ17a]

Finite lattices and monotone ML
[FPS]

MV-logics [BCM19]
Polarity-based and graph-based semantics [CFPPW]
Sahlqvist via translation [CPZ19]
Constructive canonicity [CP16, CCPZ]
Jónsson-style vs Sambin-style canonicity [PSZ17b]

Canonicity via pseudo-correspondence [CPSZ]

Constructive canonicity [CFPPW]
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Main tools of unified correspondence

**Parametric Sahlqvist theory**

- Definition of **Sahlqvist** formulas/sequents for all signatures of normal (D)LE-logics
- in terms of the order-theoretic properties of the algebraic interpretation of logical connectives

The algorithm ALBA (also **parametric**)

- computes the first-order correspondent of normal DLE-terms/inequalities.
- reduction steps sound on complex algebras of relational structures (perfect normal DLEs)
Normal DLE-logics

(D)LE: (Distributive) Lattice Expansions: \( \mathbb{A} = (\mathbb{L}, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A}) \)
(distributive) lattice signature + operations of any finite arity. Additional operations partitioned in families \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \).

**Normality**: In each coordinate,
- \( f \)-type operations preserve finite \textit{joins} or reverse finite \textit{meets};
- \( g \)-type operations preserve finite \textit{meets} or reverse finite \textit{joins}.

**Examples**
- Distributive Modal Logic: \( \mathcal{F} := \{\Diamond, \lt\} \) and \( \mathcal{G} := \{\Box, \triangleright\} \)
- Bi-intuitionistic modal logic: \( \mathcal{F} := \{\Diamond, \rightarrow\} \) and \( \mathcal{G} := \{\Box, \rightarrow\} \)
- Full Lambek calculus: \( \mathcal{F} := \{\circ\} \) and \( \mathcal{G} := \{\/, \}\)
- Lambek-Grishin calculus: \( \mathcal{F} := \{\circ, \/, \lt, \}\) and \( \mathcal{G} := \{\oplus, \lt, \}\)
- ...  

**Relational/complex algebra semantics**
- \( f \)-type operations have residuals \( f^i_\# \) in each coordinate \( i \);
- \( g \)-type operations have residuals \( g^j_h \) in each coordinate \( h \).
Inductive inequalities
Examples: reflexivity and transitivity

\[ \forall p[\Box p \leq p] \]

iff \[ \forall p \forall j \forall m[(j \leq \Box p & p \leq m) \Rightarrow j \leq m] \] (generators)

iff \[ \forall p \forall j \forall m[(\Diamond j \leq p & p \leq m) \Rightarrow j \leq m] \] (adjunction)

iff \[ \forall j \forall m[\Diamond j \leq m \Rightarrow j \leq m] \] (Ackermann)

iff \[ \forall j[j \leq \Diamond j] \] (Inv. Ackermann)

Modularity: One reduction, many translations! On Kripke frames:

\[ \forall j[j \leq \Box j] \Rightarrow \forall x[\Delta[\{x\}] \subseteq R[\{x\}] \]

i.e. \[ \Delta \subseteq R \]

\[ \forall j[3^{\Diamond j} \leq 3^j] \Rightarrow \forall x[R^{-1}[R^{-1}[\{x\}] \subseteq R^{-1}[\{x\}] \]

i.e. \[ R; R \subseteq R \]

But how about more general semantic contexts?
Examples: reflexivity and transitivity

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iff \[ \forall p \forall j \forall m[(j \leq \Box p \land p \leq m) \Rightarrow j \leq m] \] (generators)
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**Modularity:** One reduction, many translations! On Kripke frames:

\[\forall j[j \leq \Diamond j] \leadsto \forall x[\Delta[\{x\}] \subseteq R[\{x\}]\] i.e. \[\Delta \subseteq R\]
\[\forall j[\Diamond \Diamond j \leq \Diamond j] \leadsto \forall x[R^{-1}[R^{-1}[\{x\}] \subseteq R^{-1}[\{x\}]\] i.e. \[R^{-1} ; R \subseteq R\]

But how about more general semantic contexts?
Questions

Conceptual questions

▸ can we connect the meaning of the first-order correspondents in the ‘Boolean contexts’ to the meaning of those in other contexts?

▸ can we characterize or capture (or even define) meaning-preservation across contexts?

Technical questions

▸ is there an automated way to syntactically generate fist-order correspondents from those on the Boolean context?

▸ more broadly, is there an automated way to syntactically generate fist-order correspondents relative to a more general semantic context from those relative to a more restricted context?
Case Studies
CS1: Polarity-based semantics of LE-logics

Formal contexts \((A, X, I)\) are abstract representations of databases:

\[ A: \text{set of Objects} \]
\[ X: \text{set of Features} \]
\[ I \subseteq A \times X. \text{Intuitively, } alx \text{ reads: object } a \text{ has feature } x \]

Formal concepts: “rectangles” maximally contained in \(I\)
Formulas as formal concepts

Let \( P = (A, X, I) \) and \( P^+ \) be the complex algebra of \( P \).

**Models:** \( M := (P, V) \) with \( V : Prop \rightarrow P^+ \)

\[
V(p) := ([p], [p])
\]

- membership: \( M, a \triangleright p \) iff \( a \in [[p]]_M \)
- description: \( M, x > p \) iff \( x \in ([p])_M \)
Semantics of modal formulas

**Enriched formal contexts:** \( F = (A, X, I, \{ R_i | i \in \text{Agents} \}) \)

\( R_i \subseteq A \times X \) and \( \forall a ((R^\uparrow[a])^\downarrow = R^\uparrow[a]) \) and \( \forall x((R^\downarrow[x])^\uparrow = R^\downarrow[x]) \)

\( \square_i \varphi \): concept \( \varphi \) according to agent \( i \)

\[
V(\square_i \varphi) = \square_i V(\varphi) = (R^\downarrow_i[[\varphi]]), (R^\downarrow_i[[\varphi]])^\uparrow)
\]

\( \mathcal{M}, a \models \square_i \varphi \) if and only if for all \( x \in X \), if \( \mathcal{M}, x \succ \varphi \), then \( a R_i x \)

\( \mathcal{M}, x \succ \square_i \varphi \) if and only if for all \( a \in A \), if \( \mathcal{M}, a \models \square_i \varphi \), then \( aIx \)
Epistemic interpretation

Reflexivity aka Factivity
\[ \forall p[\square p \leq p] \]
iff \[ \forall j[j \leq \Diamond j] \]
iff \[ \forall a[a \uparrow \downarrow \subseteq R \downarrow[a \uparrow]] \]
iff \[ \forall a[a \in R \downarrow[a \uparrow]] \quad (R \downarrow[a \uparrow] \text{ Galois-stable}) \]
iff \[ R_i \subseteq I \] Agent i’s attributions are factually correct!

Transitivity aka Positive introspection
\[ \forall p[\square p \leq \square \square p] \]
iff \[ \forall m[\square m \leq \square \square m] \]
iff \[ \forall x[R \downarrow[x \uparrow \uparrow] \subseteq R \downarrow[(R \downarrow[x \uparrow \uparrow]) \uparrow]] \]
iff \[ \forall x[R \downarrow[x] \subseteq R \downarrow[(R \downarrow[x]) \uparrow]] \quad (R \downarrow[a \uparrow] \text{ Galois-stable}) \]
iff \[ R \subseteq R; R \]

If agent i recognizes object a as an x-object, then i must also attribute to a all the features shared by x-objects according to i.
CS2: Graph-based semantics of LE-logics

One-sorted structures $\mathcal{G} = (Z, E)$, with $E$ reflexive:

$\mathcal{G}^+ := (Z, Z, E^c)^+$
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Representation. States: maximally disjoint filter-ideal pairs $(F, I)$;

$$(F, I) E (F', I') \iff F \cap I' = \emptyset$$
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One-sorted structures $\mathcal{G} = (Z, E)$, with $E$ reflexive:

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Representation. States: maximally disjoint filter-ideal pairs $(F, I)$;

$(F, I) E (F', I')$ iff $F \cap I' = \emptyset$

$M, z \Vdash \square \psi$ iff $\forall z' [z R \square z' \Rightarrow M, z' \not\Vdash \psi]$

$M, z \succ \square \psi$ iff $\forall z' [z' E z \Rightarrow M, z' \not\Vdash \square \psi]$
Modelling informational entropy

**Informational entropy**: an inherent boundary to knowability, due e.g. to perceptual, theoretical, evidential or linguistic limits.

**Reflexivity as** $E$-**reflexivity**

$$\forall p[\square p \leq p]$$

iff $$\forall j[j \leq \Diamond j]$$

iff $$\forall z[z^{[10]} \subseteq R^{[0]}[z^{[1]}]]$$

iff $$E \subseteq R$$

the agent correctly recognizes inherent indistinguishability

**Transitivity as** $E$-**transitivity**

$$\forall p[\square p \leq \boxdot \square p]$$

iff $$\forall j[\Diamond \Diamond j \leq \Diamond j]$$

iff $$\forall z[R^{[0]}[(R^{[0]}[z^{[01]}])[1]]] [1]) \subseteq R^{[0]}[z^{[01]}]$$

iff $$R \circ E R \subseteq R$$

**$E$-compositions** of $R, S \subseteq Z \times Z$:

$$x(R \circ_E S)a \text{ iff } \exists b(xRb & E^{(1)}[b] \subseteq S^{(0)}[a]).$$

$$a(R \Diamond_E S)x \text{ iff } \exists y(aRy & E^{(0)}[y] \subseteq S^{(0)}[x]).$$
Truth-value space: A finite (or complete, or perfect) Heyting algebra $\mathbb{A}$.

Formulas of $L_\mathbb{A}$:

$$\varphi ::= t | p | \varphi \lor \psi | \varphi \land \psi | \varphi \rightarrow \psi | \Box \varphi | \Diamond \varphi$$

Models $\mathcal{M} = (W, R, V)$, where

- $W \neq \emptyset$
- $R : W \times W \to \mathbb{A}$
- $V : (\text{Prop} \times W) \to \mathbb{A}$.

Semantics

- $V(t, w) = t \in A$
- $V(\Diamond p, w) = \bigvee^{\mathbb{A}} \{Rwu \land^{\mathbb{A}} V(p, u) | u \in W\}$
- $V(\Box p, w) = \bigwedge^{\mathbb{A}} \{Rwu \rightarrow^{\mathbb{A}} V(p, u) | u \in W\}$
Correspondence theory for MV-modal logic

This is work of Britz, Conradie, and Morton.
Let $A$ be a perfect Heyting algebra and $a \in A$.

Theorem
Every inductive formula has a effectively computable local frame $a$-correspondent of the class of $A$-frames.

Corollary
Every Sahlqvist formula has an effectively computable local frame $a$-correspondent of the class of $A$-frames.
A preservation result

This is work of Britz, Conradie, and Morton.

Restricted Sahlqvist formulas

A restricted Sahlqvist implication is an implication $\varphi \rightarrow \psi$ in which

1. $\varphi$ is built from boxed atoms ($\Box^n p$) by applying $\land$, $\lor$ and $\Diamond$.
2. $\psi$ is positive.
3. For each $p \in Prop$ in $\psi$, $p$ does not occur in any subformula $\alpha$ such that $\alpha \rightarrow \gamma$ is a subformula of $\varphi$.

A restricted Sahlqvist formula is built from restricted Sahlqvist implications by applying $\land$ and $\Box$.

Theorem

Let $\varphi$ be a restricted Sahlqvist formula and let $\alpha$ be its classical local frame correspondent. Then:

$$\mathcal{G}, w \Vdash_a \varphi \rightarrow \psi \quad \text{iff} \quad \mathcal{G} \models_a \alpha[x := w]$$
Example: $a$-validity of reflexivity $p \rightarrow \lozenge p$

\[
\forall p[a \leq p \rightarrow \lozenge p] \\
\text{iff } \forall p[p \land a \leq \lozenge p] \\
\text{iff } \forall p \forall i \forall m[(i \leq p \land a \land \lozenge p \leq m) \Rightarrow i \leq m] \\
\text{iff } \forall p \forall i \forall m[(i \leq a \land i \leq p \land \lozenge p \leq m) \Rightarrow i \leq m] \\
\text{splitting} \\
\text{iff } \forall i \forall m[(i \leq a \land \lozenge i \leq m) \Rightarrow i \leq m] \\
\text{Ackermann} \\
\text{iff } \forall i[i \leq a \Rightarrow i \leq \lozenge i] \\
\text{inv. Ackermann} \\
i.e. \Delta \subseteq R \text{ relativized to } a.
Preliminary conclusions

- Notation, notation, notation.
- From parametricity to modularity and back.
- Both syntactic and semantic parameters.
- Preservation of syntactic shape but not of meaning; preservation of meaning but not of syntactic shape.
- Is there more than an optical illusion?