Duality for Algebras of the Connected Planar Wiring Diagrams Operad

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Combining Resistance

\[ R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2} \]

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Combining Resistance

\[ R_1 + R_2 \]

\[ \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \]
Conductance is Inverse Resistance

\[ G = \frac{1}{R} \]

\( R = 2 \, \Omega \) \quad \Rightarrow \quad \text{conductance} \quad \sim \sim \quad G = 0.5 \, \Omega

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Combining Conductance

\[ G_1 \approx G_2 \approx (\frac{1}{G_1} + \frac{1}{G_2})^{-1} \]

\[ G_1 + G_2 \]
Combining Maximum Flow Rates

\[ F_1 \sim \min(F_1, F_2) \]

\[ F_1 \sim F_1 + F_2 \]

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Combining Minimum Path Lengths

\[ D_1 + D_2 = \min(D_1, D_2) \]
## Series and parallel formulas

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Series and parallel connections are dual

Not quite the usual "dual graph."
Connected circular planar graphs
Placing the dual vertices
Placing the dual edges
The dual graph
Dual Connected Circular Planar Graphs
Resistance and conductance are “dual”

**Resistances**

\[
\text{Effective resistance} = \frac{157}{29} \, \Omega
\]

**Conductances**

\[
\text{Effective conductance} = \frac{157}{29} \, \Omega
\]
Max flow rate and min path length are “dual”

Max flow rates

Overall max flow rate = 3

Edge lengths

Min path length = 3
Main theorem

Theorem (B.—, 2019)

- (Connected circular planar) graphs form an algebra of the operad \( \mathcal{Plan} \) of “connected planar wiring diagrams.”
- “Max flow rate,” “min path length,” “effective resistance,” and “effective conductance” are all morphisms between \( \mathcal{Plan} \)-algebras.
- \( \mathcal{Plan} \) has a duality automorphism giving every algebra a “dual” algebra.
- The algebra of graphs is isomorphic to its dual algebra, and the isomorphism sends a graph to its dual graph.
- The dual of the “max flow rate” morphism is “min path length,” and the dual of “effective resistance” is “effective conductance.”
Gluing together circular planar graphs
A connected planar wiring diagram

This is a *morphism* in the operad *Plan* of connected planar wiring diagrams.
Series and parallel wiring diagrams

Series

Parallel
Definition (B.—, 2019)

The (symmetric, coloured) operad $\mathcal{Plan}$:
- objects are circularly ordered finite sets $(X, \theta)$.
The operad of connected planar wiring diagrams

Definition (continued)

- morphisms from \((X_1, \theta_1), \ldots, (X_n, \theta_n)\) to \((Y, \varphi)\) are planar wiring diagrams with the \(X_i\) on the inside and \(Y\) on the outside:

Every “cable” has \(\leq 1\) element of \(Y\).

Every “face” has \(\leq 1\) arc of outer circle.

Lemma: This really does define an operad!
Plan has:

- A forgetful map to Spivak’s operad of all wiring diagrams. Plan inherits several algebras from there, like flows and potentials. However, Spivak’s operad does not have a duality automorphism.

- An inclusion map to Jones’s “planar algebras” operad. Plan inherits its duality automorphism Jones’s “1-click” automorphism, but Jones’s operad has too many morphisms for circular planar graphs to be an algebra.
A $\mathcal{P}lan$-algebra $\mathcal{A}$ assigns:

- to each circularly ordered set $(X, \theta)$ a set $\mathcal{A}(X, \theta)$, and
- to each morphism $(X_1, \theta_1), \ldots, (X_n, \theta_n) \rightarrow (Y, \varphi)$ a function

$$\prod_{i=1}^{n} \mathcal{A}(X_i, \theta_i) \rightarrow \mathcal{A}(Y, \varphi).$$

These describe $what$ may be inserted into the slots of a wiring diagram and $how$ they glue together.

$$\mathcal{P}lan \xrightarrow{\mathcal{A}} \text{Op}(\text{Set})$$
Example Plan algebras: $\mathcal{G}$ and $\mathcal{G}_{(0,\infty)}$

- $\mathcal{G}(X, \theta) =$ set of connected circular planar graphs with boundary vertices $(X, \theta)$.
- $\mathcal{G}_{(0,\infty)}(X, \theta)$: same, but with edges weighted by positive real numbers.

Lemma: $\mathcal{G}$ is generated by the single element

\[\text{Diagram:}\]

satisfying a single relation. That makes it easy to describe algebra morphisms out of $\mathcal{G}$!
Example *Plan* algebra: $\Pi$

- $\Pi(X, \theta) =$ set of planar (noncrossing) partitions of $(X, \theta)$. 
**Example Plan-algebra: potential sets**

- A potential on \((X, \theta)\) is a function \(X \rightarrow \mathbb{R}\) up to overall additive constant:

\[
\begin{align*}
0 &\quad 1 \\
-2 &\quad 4 \\
2 &\quad 100 \\
98 &\quad 104 \\
102 &\quad 101
\end{align*}
\]

\(\mathcal{V}(X, \theta)\) is the set of potentials on \(X\).
Gluing compatible potentials

\[ \text{Diagram with labeled nodes and connections.} \]
Gluing compatible potentials

\[ 3^2 \overset{1}{\underset{-1}{\to}} 3 \]

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Gluing potential sets

Not all potentials can be glued, so $\mathcal{V}$ is only a \textit{relational} Plan-algebra. But

$$\mathcal{P}(\mathcal{V}) : (X, \theta) \mapsto \text{the set of subsets of } \mathcal{V}(X, \theta)$$

is an actual Plan-algebra!

Send a collection of potential sets to the set of potentials obtained by gluing compatible members.
Min path length: $G_{(0, \infty)} \rightarrow \mathcal{P}(\mathcal{V})$

Weights on a graph $\sim$ distances $\sim$ potential set:

$$\{(a, b) \in \mathbb{R}^2 \mid |a - b| \leq D\}$$
Min path length: $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V})$

\[ D_1 \overset{\sim}{\to} \{ (a, b) \in \mathbb{R}^2 \mid \exists x \in \mathbb{R} \text{ such that} \]
\[ |a - x| \leq D_1, |x - b| \leq D_2, |x - b| \leq D_3 \}
\[ = \{ (a, b) \in \mathbb{R}^2 \mid |a - b| \leq D_1 + \min(D_2, D_3) \}\]

“Min path length” is an algebra morphism $\mathcal{G}_{(0,\infty)} \to \mathcal{P}(\mathcal{V})$!
The algebra $\mathcal{P}(\mathcal{F})$ of flow sets: A flow on $(X, \theta)$ is a sum-zero function $X \to \mathbb{R}$:

$$\mathcal{F}(X, \theta) = \text{the set of flows on } X.$$
Gluing compatible flows

\[-1 \quad 3 \quad -1 \quad 1 \quad -2 \quad 1 \quad -1 \quad 1\]

\[-1 \quad 1 \quad -2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -2 \quad 1 \quad 2 \quad -1 \quad 0 \quad 1 \]

\[= \]

Duality for Planar Wiring Diagrams

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Not all flows can be glued, so $\mathcal{F}$ is only a \textit{relational Plan}-algebra. But

$$\mathcal{P}(\mathcal{F}) : (X, \theta) \mapsto \text{the set of subsets of } \mathcal{F}(X, \theta)$$

is an actual \textit{Plan}-algebra!

Send a collection of flow sets to the set of flows obtained by gluing compatible members.
Max flow rate: $G_{(0,\infty)} \rightarrow \mathcal{P}(\mathcal{F})$

Weights on a graph $\rightsquigarrow$ possible flows:

\[ \{(a, -a) \in \mathbb{R}^2 \mid |a| \leq F\} \]
Max flow rate: $G_{(0,\infty)} \rightarrow \mathcal{P}(\mathcal{F})$

```
\begin{center}
\begin{tikzpicture}
    \node (A) at (0,0) {\textbullet}
    \node (B) at (1,0) {\textbullet}
    \node (C) at (2,0) {\textbullet}
    \draw[dashed] (A) to (B) node[midway, right] {$F_1$} node[midway, left] {$F_3$}
    \draw (B) to (C) node[midway, above] {$F_2$}
    \end{tikzpicture}
\end{center}

\[
\begin{array}{c}
\{(a, b) \in \mathbb{R}^2 \mid \exists x, y, z \in \mathbb{R} \text{ such that} \\
|x| \leq F_1, \ |y| \leq F_2, \ |z| \leq F_3, \\
x = a, \ y + z = b, \ x + y + z = 0 \}
\end{array}
\]

\[
= \{(a, -a) \in \mathbb{R}^2 \mid |a| \leq \min(F_1, F_2 + F_3) \}
\]

“Max flow rate” is a Plan-algebra morphism $G_{(0,\infty)} \rightarrow \mathcal{P}(\mathcal{F})$!
Resistance: $\mathcal{G}_{(0,\infty)} \rightarrow \mathcal{P}(\mathcal{V} \times \mathcal{F})$

Weights on a graph $\leadsto$ voltage-current relationships:

$$\leadsto \left\{ \left( \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} , \begin{bmatrix} I \\ -I \end{bmatrix} \right) \in \mathbb{R}^2 \times \mathbb{R}^2 \bigg| (V_1 - V_2) = IR \right\}$$

In general, send a weighted graph to the set of pairs (boundary voltages, induced boundary currents). “Effective resistance” is a $\mathcal{P}lan$-algebra morphism $\mathcal{G}_{(0,\infty)} \rightarrow \mathcal{P}(\mathcal{V} \times \mathcal{F})$!
$\text{Plan}$ has a “duality” automorphism

\[ * : \text{Plan} \rightarrow \text{Plan} \]

Compose with any algebra $A : \text{Plan} \rightarrow \text{Op}(\text{Set})$ to get a “dual” algebra

\[ A^* : \text{Plan} \xrightarrow{\ast} \text{Plan} \xrightarrow{A} \text{Op}(\text{Set}). \]

\[ A^{**} \cong A. \]
Duality automorphism of Plan: on objects

\[ \star : X \leftrightarrow X^* \]
\[ \star : Y \leftrightarrow Y^* \]
Duality automorphism of Plan: on morphisms
Duality automorphism of *Plan*: on morphisms
Duality automorphism of Plan: on morphisms
Duality automorphism of \textit{Plan}: on morphisms
Series and parallel wiring diagrams are dual
Dual of $\mathcal{G}$ is $\mathcal{G}$

$\mathcal{G}$ is isomorphic to its own dual: the isomorphism sends a graph to its dual.

The same holds for $\mathcal{G}_{(0,\infty)}$. 
Theorem (B.—, 2019)

The dual of the algebra of potentials is the algebra of flows: \( \mathcal{V}^* \cong \mathcal{F} \).

Take successive differences of potential values to obtain a flow:

\[ \begin{align*} 101 & \leftrightarrow 102 & -1 \\ 105 & \leftrightarrow 103 & 4 \\ & & \end{align*} \]

Corollary: \( \mathcal{P}(\mathcal{V})^* \cong \mathcal{P}(\mathcal{F}) \) and \( \mathcal{P}(\mathcal{V} \times \mathcal{F})^* \cong \mathcal{P}(\mathcal{V} \times \mathcal{F}) \) as well.
The dual of the “min path length” morphism is the “max flow rate” morphism: the square

\[
\begin{array}{c}
\mathcal{G}_{(0,\infty)}^{*} \quad \cong \quad \mathcal{G}_{(0,\infty)} \\
\downarrow \quad \downarrow \\
\mathcal{P}(\mathcal{V})^{*} \quad \cong \quad \mathcal{P}(\mathcal{F})
\end{array}
\]

commutes.
Dual of min path length is max flow rate

Proof sketch: only have to check for single weighted edges!

\[ \{(a, a) : |a| \leq D\} = \{(a, b) : |a - b| \leq D\} \leftrightarrow \{(a - b, b - a) : |a - b| \leq D\} \]
The dual of the “effective resistance” morphism is “effective 1/resistance”: the square

\[
\begin{array}{ccc}
G^*(0,\infty) & \overset{\cong}{\to} & G(0,\infty) \\
\downarrow & & \downarrow \\
\mathcal{P}(V \times F)^* & \overset{\cong}{\to} & \mathcal{P}(V \times F)
\end{array}
\]

commutes, where the isomorphism \( G^*(0,\infty) \cong G(0,\infty) \) dualizes the graph and inverts the edge weights.
Dual of resistance is conductance

Proof sketch:

\[ R \leadsto \left\{ \left( \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \begin{bmatrix} I \\ -I \end{bmatrix} \right) \mid (V_1 - V_2) = IR \right\} \]

\[ \leftrightarrow \left\{ \left( \begin{bmatrix} c + I \\ V_1 - V_2 \end{bmatrix}, \begin{bmatrix} V_2 - V_1 \end{bmatrix} \right) \mid (V_1 - V_2) = IR \right\} \]

\[ = \]

\[ \frac{1}{R} \leadsto \left\{ \left( \begin{bmatrix} V'_1 \\ V'_2 \end{bmatrix}, \begin{bmatrix} I' \\ -I' \end{bmatrix} \right) \mid (V'_1 - V'_2) = I'/R \right\} \]
Thank you!