Circuit Relations for Real Stabilizers: Towards TOF + $H$

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Outline

1. Graphical calculi for circuits
2. Classical reversible computing
3. Quantum computing
4. Classical reversible and quantum computing
The graphical calculus for PROPs models circuit computation.

Finite presentations of different fragments of computing are studied.

Complete presentation is a strict $\dagger$-symmetric monoidal faithful functor.

For quantum computing: ZX-calculus, ZH-calculus, ZW-calculus.

For reversible computing: CNOT and TOF.
The category CNOT

Consider the PROP generated by cnot, $|1\rangle$, $\langle 1|$:

The controlled not gate, cnot, takes bits:

$$|b_1, b_2\rangle \mapsto |b_1, b_1 \oplus b_2\rangle$$

cnot is drawn as:

$|1\rangle$ is preparing 1 and $\langle 1|$ is postselecting 1:

The not gate and $|0\rangle$, $\langle 0|$ are derived:

This category has a finite, complete presentation in terms of *circuit identities*, CNOT [CCS18]:
The identities of CNOT

[CNOT.1] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot1.png}}
\end{array}
\]

[CNOT.2] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot2.png}}
\end{array}
\]

[CNOT.3] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot3.png}}
\end{array}
\]

[CNOT.4] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot4.png}}
\end{array}
\]

[CNOT.5] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot5.png}}
\end{array}
\]

[CNOT.6] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot6.png}}
\end{array}
\]

[CNOT.7] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot7.png}}
\end{array}
\]

[CNOT.8] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot8.png}}
\end{array}
\]

[CNOT.9] = \[
\begin{array}{c}
\text{\includegraphics[width=1cm]{cnot9.png}}
\end{array}
\]
The category TOF

Consider the PROP generated by tof, $|1\rangle$, $\langle 1|$: 

The Toffoli gate, tof, takes bits:

$$|b_1, b_2, b_3\rangle \mapsto |b_1, b_1 \cdot b_2 \oplus b_1\rangle$$

tof is drawn as:

$|1\rangle$ is preparing 1 and $\langle 1|$ is postselecting 1:

The cnot gate is derived:

This category has a finite, complete presentation, TOF [CC19]:
The identities of TOF

[TOF.1]

[TOF.2]

[TOF.3]

[TOF.4]

[TOF.5]

[TOF.6]

[TOF.7]
The identities of TOF

\begin{align*}
\text{[TOF.8]} & \quad = \quad \text{[TOF.13]} \\
\text{[TOF.9]} & \quad = \quad \text{[TOF.14]} \\
\text{[TOF.10]} & \quad = \quad \text{[TOF.15]} \\
\text{[TOF.11]} & \quad = \quad \text{[TOF.16]} \\
\text{[TOF.12]} & \quad = \quad \text{[TOF.16]} \\
\end{align*}
Both CNOT and TOF have concrete equivalent categories. In particular they are discrete inverse categories.

That means that they have a total copying map generated by:
Frobenius algebras

A Frobenius algebra is a monoid-comonoid pair:

Satisfying the Frobenius law:
Frobenius algebras

A Frobenius algebra is **commutative** if:

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) circle (0.1cm);
\filldraw (1,0) circle (0.1cm);
\draw (-0.5,0) -- (1.5,0);
\draw (0,0) to [out=90,in=180] (0,1);
\draw (0,0) to [out=0,in=90] (0,-1);
\draw (0,1) to [out=270,in=180] (1,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) circle (0.1cm);
\filldraw (1,0) circle (0.1cm);
\draw (-0.5,0) -- (1.5,0);
\draw (0,0) to [out=90,in=180] (0,1);
\draw (0,0) to [out=0,in=90] (0,-1);
\draw (0,1) to [out=270,in=180] (1,0);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) circle (0.1cm);
\filldraw (1,0) circle (0.1cm);
\draw (-0.5,0) -- (1.5,0);
\draw (0,0) to [out=90,in=180] (0,1);
\draw (0,0) to [out=0,in=90] (0,-1);
\draw (0,1) to [out=270,in=180] (1,0);
\end{tikzpicture}
\end{array}
\end{align*}

And **special** if:

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) circle (0.1cm);
\filldraw (0,0.5) circle (0.1cm);
\filldraw (0,-0.5) circle (0.1cm);
\draw (-0.5,0) -- (0.5,0);
\draw (0,0) .. controls (0,0.5) and (0,0.5) .. (0,0.5);
\draw (0,0) .. controls (0,-0.5) and (0,-0.5) .. (0,-0.5);
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) circle (0.1cm);
\filldraw (0,0.5) circle (0.1cm);
\filldraw (0,-0.5) circle (0.1cm);
\draw (-0.5,0) -- (0.5,0);
\draw (0,0) .. controls (0,0.5) and (0,0.5) .. (0,0.5);
\draw (0,0) .. controls (0,-0.5) and (0,-0.5) .. (0,-0.5);
\end{tikzpicture}
\end{array}
\end{align*}

Connected components of Frobenius algebras can be uniquely represented by spiders:
Theorem ([CPV13])

Orthonormal bases \( \{ |i\rangle \}_{i \in B} \) in \( \text{FdHilb} \) are in one-to-one correspondence with special, commutative \( \dagger \)-Frobenius algebras:

\[
\sum_{i \in B} |i\rangle \quad \sum_{i \in B} |i\rangle \langle i, i | \quad \sum_{i \in B} \langle i | \quad \sum_{i \in B} |i, i\rangle \langle i |
\]

Therefore, we can consider the Frobenius algebras associated to the eigenbasis of quantum observables. For example, consider the Hermetian matrices:

\[
X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

\( X \) and \( Z \) have spectra:

\[
X_+ = |+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle) \quad X_- = |-\rangle = 1/\sqrt{2}(|0\rangle - |1\rangle)
\]

\[
Z_+ = |0\rangle, \quad Z_- = |1\rangle
\]
The **phase-free ZX-calculus**, \(ZX_\pi\), [DP13] is the PROP generated by the \(Z\) Frobenius algebra and Hadamard gate:

![ ZX-calculus diagrams ]

The Hadamard gate is a self-inverse change of basis matrix so that:

\[
X_+ H = Z_+ H \quad X_- H = Z_- H \quad Z_+ H = X_+ H \quad Z_- H = X_- H
\]

The Frobenius algebra for \(X\) is therefore given by conjugation.

![ Frobenius algebra for X diagrams ]
$\mathbf{ZX}_\pi$ has a finite presentation:

The first identity is that *the axioms of a special†-Frobenius algebra hold for Z*.

The Frobenius algebras associated to the Z and X observables are strongly complimentary.

They form a Hopf algebra up to an invertible scalar:

$[\text{B.U’}] = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}$

$[\text{B.M’}] = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}$

$[\text{B.H’}] = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}$

This corresponds to the bases being mutually unbiased [CD11].
Consider the extension of CNOT with the Hadamard gate (and $\sqrt{2}$):

\[
\begin{align*}
[H.I] & \quad = \\
[H.L] & \quad =
\end{align*}
\]

\[
\begin{align*}
[H.F] & \quad = \\
[H.Z] & \quad =
\end{align*}
\]

\[
\begin{align*}
[H.U] & \quad = \\
[H.S] & \quad =
\end{align*}
\]
Consider $G : ZX_\pi \rightarrow \text{CNOT} + H$, sending:

\[\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\text{Original} \\
ZX_\pi
\end{array} & \rightarrow & \\
\begin{array}{c}
\text{CNOT + H}
\end{array} \\
\begin{array}{c}
\text{Original} \\
\text{CNOT + H}
\end{array}
\end{array}
\end{align*}\]
Consider $F : \text{CNOT} + H \rightarrow ZX_{\pi}$, sending:

- $\text{CNOT} + H \rightarrow ZX_{\pi}$
- $\sqrt{2} \rightarrow \bigcirc \bigcirc$
- $\pi \rightarrow \bigcirc \bigcirc$
- $\pi \rightarrow \bigcirc \bigcirc$
- $\bigcirc \bigcirc \rightarrow \bigcirc \bigcirc$
- $\bigcirc \bigcirc \rightarrow \bigcirc \bigcirc$
- $\bigcirc \bigcirc \rightarrow \bigcirc \bigcirc$
- $\bigcirc \bigcirc \rightarrow \bigcirc \bigcirc$
CNOT is isomorphic to $\text{ZX}_\pi$

**Proposition**

$F : \text{CNOT} + H \rightarrow \text{ZX}_\pi$ and $G : \text{ZX}_\pi \rightarrow \text{CNOT} + H$ are $\dagger$-preserving symmetric monoidal functors.

**Theorem**

$F : \text{CNOT} + H \rightarrow \text{ZX}_\pi$ and $G : \text{ZX}_\pi \rightarrow \text{CNOT} + H$ are inverses. This implies that $\text{CNOT} + H$ is complete...

**Theorem ([DP13])**

$\text{ZX}_\pi$ is complete for real stabilizer circuits.

We can also remove the scalar $\sqrt{2}$ by being careful.
Stabilizer circuits and universality

There is a caveat:

**Theorem ([Got98])**

*Stabilizer circuits can be simulated in polynomial time on a classical probabilistic computer.*

However,

**Theorem ([Aha03])**

*The Toffoli and Hadamard gates, together are an approximately universal gate set for quantum computing.*
Is there a presentation in terms of the Toffoli gate and $H$?

But
The Toffoli gate has the following representation in $\text{ZX}_\pi$ [NW18]:

\[ \pi \quad \pi \]

However, the Triangle has the following representation in CNOT + $H$ [Vil18]:

\[ \sqrt{2} \]

\[ \sqrt{2} \]
The ZH-calculus

The controlled-Z gate can be represented with Toffoli gate and Hadamard:

\[
\begin{array}{c}
\quad \\
\end{array}
\]

In the ZH calculus controlled Z-gates are given by:

Axiom \([H.F]\) of CNOT + \(H\) generalizes to Toffoli gates:

\[
[H.F']
\]
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