Equivalence between Orthocomplemented Quantales and Complete Orthomodular Lattices.

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Hilbert spaces are popular for reasoning about quantum theory, but in many ways extraneous (quantum states are one-dimensional subspaces, abstracting away individual vectors).

Different simpler quantum structures highlight different aspects of quantum reasoning:

- **Complete orthomodular lattice**: ortholattice of testable properties gives a *static* perspective
- **Orthomodular dynamic algebra**: quantale of quantum actions enriched with an orthogonality operator gives *dynamic* perspective

A categorical equivalence between these structures clarifies how these perspectives are related.
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A complete orthomodular lattice

A structure \((L, \leq, -\perp)\) such that

- \((L, \leq)\) is a complete lattice (has arbitrary joins)
- \(\perp\) is a lattice orthocomplement:
  - \(\perp\) is a complement: \(a \land a\perp = O\) and \(a \lor a\perp = I\).
  - \(\perp\) is involutive: \((a\perp)\perp = a\)
  - \(\perp\) is order reversing: \(a \leq b\) implies \(b\perp \leq a\perp\).
- orthomodular (weakened distributivity) law holds: \(q \leq p\) implies \(p \land (p\perp \lor q) = q\).

Example (Hilbert lattice)

closed subspaces of a Hilbert space.

The points of lattice are quantum testable properties.
What about dynamics?

Sasaki hook and projection

Given testable properties $p, q$

- $f^p(q) \overset{\text{def}}{=} p^\perp \lor (p \land q)$ (hook)
  The precondition of a projection onto $p$ resulting in $q$

- $f_p(q) \overset{\text{def}}{=} p \land (p^\perp \lor q)$ (projection)
  The result of projecting $q$ onto $p$
A quantale ("quantum locale") is a tuple $(Q, \sqsubseteq, \cdot)$, such that

- $(Q, \sqsubseteq)$ is sup-lattice (complete lattice)
- $(Q, \cdot)$ is a monoid satisfying the following distributive laws

\[
\begin{align*}
a \cdot \bigsqcup S &= \bigsqcup \{a \cdot b \mid b \in S\} \\
\bigsqcup S \cdot a &= \bigsqcup \{b \cdot a \mid b \in S\}
\end{align*}
\]

Quantales relate to operator algebras: the points of a quantale can be thought of as operators on a Hilbert space.

Temporal meaning from monoidal composition:

$a \cdot b$ read "$a$ after $b$" (quantum observables are not commutative)
An application: dynamics acting on states

- **$Q$ - a quantale** (a set with certain algebraic structure)
  Elements of $Q$: nondeterministic “actions” or “observations”

- **$M$ - module** over $Q$
  Elements of $M$: nondeterministic “states” or “processes”

- $\star : Q \times M \rightarrow M$
  “action” of quantale $Q$ on module $M$

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Baltag and Smets introduce a Quantum dynamic algebra: A quantale augmented with an orthogonality operator $\sim$.


We modify their definition to ensure categorical equivalences with complete orthomodular lattices.
A quantum dynamic algebra is a type of generalized dynamic algebra.

**Definition (Generalized dynamic algebra)**

A generalized dynamic algebra is a tuple $\mathcal{Q} = (Q, \sqcup, \cdot, \sim)$, such that

- $Q$ is a set of quantum actions (typically infinite)
- $\sqcup : \mathcal{P}(Q) \rightarrow Q$ (for choice),
- $\cdot : Q \times Q \rightarrow Q$ (for sequential observation or action)
- $\sim : Q \rightarrow Q$ (similar to an orthocomplement)
Generalized dynamic algebra concepts

Given a generalized dynamic algebra $\mathcal{Q} = (Q, \sqcup, \cdot, \sim)$

$$(x \sqsubseteq y) \text{ iff } (x \sqcup y = y)$$

Potential lattice of “projectors” inside $\mathcal{Q}$:

$$\mathcal{P}_\mathcal{Q} \overset{\text{def}}{=} \{ \sim x \mid x \in Q \}$$

$$\bigvee X \overset{\text{def}}{=} \sim \sim \sqcup X \quad \text{for all } X \subseteq \mathcal{P}_\mathcal{Q}$$

$$\bigwedge X \overset{\text{def}}{=} \sim \sqcup \sim X \quad \text{for all } X \subseteq \mathcal{P}_\mathcal{Q}$$

$$A \preceq B \iff A \land B = A \quad \text{for all } A, B \in \mathcal{P}_\mathcal{Q}$$

Observed action and equivalence:

$$\llbracket x \rrbracket \overset{\text{def}}{=} \lambda y.\sim\sim(x \cdot y)$$

$$x \equiv y \iff \llbracket x \rrbracket(p) = \llbracket y \rrbracket(p) \text{ for all } p \in \mathcal{P}_\mathcal{Q}$$

Potential “atoms” of $\mathcal{Q}$ built from $\mathcal{P}_\mathcal{Q}$.

• $\mathcal{T}_\mathcal{Q}$ is the smallest superset of $\mathcal{P}_\mathcal{Q}$ closed under composition
Concrete example: a Hilbert space realization

\( \mathcal{H} \) - Hilbert space
\( \mathcal{P}_\mathcal{H} \) - the set of singleton sets of projectors \( P_A \) onto closed linear subspaces \( A \).

Example

\( \mathcal{Q} = (Q, \bigcup, \cdot, \sim) \), where

- \( Q = \mathcal{P}(\mathcal{T}_\mathcal{H}) \) where \( \mathcal{T}_\mathcal{H} \) is the smallest superset of \( \mathcal{P}_\mathcal{H} \) closed under composition. (An element of \( Q \) is a set)
- \( \bigcup \) is just the union operation
  (union of sets of functions, not unions of functions)
- \( \cdot \) is defined by \( A \cdot B = \{a \circ b \mid a \in A, b \in B\} \)
  (function composition of each pair of functions)
- \( \sim \) is defined by \( \sim A = \{P_{B^\perp}\} \) where \( B = \text{Im}(\bigcup_{a \in A} a) \).
Quantale inside our Hilbert space realization

The Hilbert space realization satisfies:

- \((Q, \sqsubseteq, \cdot)\) is a quantale:
  - \((Q, \sqsubseteq)\) is a complete lattice
  - \((Q, \cdot)\) is a monoid, where

\[
\begin{align*}
  a \cdot \bigsqcup S &= \bigsqcup \{a \cdot b \mid b \in S\} \\
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- \(\mathcal{P}_\Omega = \mathcal{P}_\mathcal{H}\)
- \(\mathcal{T}_\Omega = \mathcal{T}_\mathcal{H}\).

- \((\mathcal{P}_\Omega, \preceq, \sim)\) is a Hilbert lattice, and hence a complete orthomodular lattice.

The orthogonality operator \(\sim\) is not a lattice orthocompletent for the quantale lattice, but for the induced lattice \((\mathcal{P}_\Omega, \preceq, \sim)\).
Orthomodular dynamic algebra (ODA)

A *generalized dynamic algebra* $\mathcal{Q} = (Q, \sqcup, \cdot, \sim)$ is an *orthomodular dynamic algebra* if for all $p, q \in \mathcal{P}_Q$, $x, y \in \mathcal{T}_Q$, and $X, Y \subseteq \mathcal{T}_Q$:

1. $(Q, \sqsubseteq, \cdot)$ is a quantale and $\sqcup$ is its arbitrary join.
2. $(\mathcal{P}_Q, \preceq, \sim)$ is a complete orthomodular lattice
3. $Q$ is generated from $\mathcal{P}_Q$ by $\cdot$ and $\sqcup$ (*minimality*)
   (ensures $Q$ does not have too many elements.)
4. $x = y$ iff $x \equiv y$ (*completeness*)
   (ensures distinct behavior of distinct elements.)
5. $\sqcup X = \sqcup Y$ iff $X = Y$ (*atomicity*)
6. $\overline{\overline{p}(q)} = f_p(q)$ (i.e. $\sim\sim(p \cdot q) = p \land (\sim p \lor q)$) (*Sasaki projection*)
   (connects monoidal to orthomodular lattice dynamics)
7. $\overline{\overline{x}(y)} = \overline{\overline{x}(\sim\sim y)}$ (*composition*)
   ($\overline{\overline{x}}$ acting on $Q$ is fully determined by its action on $\mathcal{P}_Q$)
Let $\mathbb{L}$ be the category with

**Object:** Complete orthomodular lattices

**Morphisms:** Ortholattice isomorphisms:

- Bijections $k$ preserving order and orthocomplementation:
  - $p \leq_1 q$ if and only if $k(p) \leq_2 k(q)$
  - $k(p^{\perp 1}) = (k(p))^{\perp 2}$. 
Let $\mathcal{Q}$ be the category with

**Objects**: Orthomodular dynamic algebras

**Morphisms**: Functions $\theta: \mathcal{Q} \rightarrow \mathcal{R}$ satisfying:

- $\theta$ preserves $\cdot$, $\bigvee$.
- The restriction of $\theta$ to $\mathcal{P}_\mathcal{Q}$ (the image of $\mathcal{Q}$ under $\sim$) is on ortholattice isomorphism (hence maps $\mathcal{P}_\mathcal{Q}$ to $\mathcal{P}_\mathcal{R}$)
Definition (Categorical Equivalence)

An equivalence between categories \( \mathbb{L} \) and \( \mathbb{Q} \) is a pair of covariant functors

\[
(F : \mathbb{L} \to \mathbb{Q}, U : \mathbb{Q} \to \mathbb{L})
\]

such that

1. there is a natural isomorphism \( \eta : 1_{\mathbb{Q}} \to F \circ U \)
2. there is a natural isomorphism \( \tau : 1_{\mathbb{L}} \to U \circ F \)
Translation $F : \mathbb{L} \rightarrow \mathbb{Q}$ from lattice to algebra

on objects

Let $\mathcal{L} = (L, \leq, -\perp)$ be a complete orthomodular lattice. Define

$$F_T = \text{smallest set containing } \{f_p \mid p \in L\},$$

closed under composition

$$Q = \mathcal{P}(F_T)$$

$$A \cdot B = \{f \circ g \mid f \in A, g \in B\}$$

$$\sim A = \bigvee\{a(I) \mid a \in A\}, \quad \text{(where } I = \bigwedge \emptyset \text{ is the top element)}$$

Then $F(\mathcal{L}) = (Q, \cdot, \sim)$

on morphisms

If $k : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a morphism (ortholattice isomorphism), then

$F(k) : A \rightarrow \{k \circ a \circ k^{-1} \mid a \in A\}$ conjugates every element of input $A$ by $k$. 
A useful property: preservation of projectors

If \( p \in L_1 \), then \( k \circ f_p \circ k^{-1} = f_{k(p)} \).

**Proof.**

For \( b \in L_2 \),

\[
\psi_k(f_p)(b) = k \circ f_p \circ k^{-1}(b) \\
= k(p \land (p \perp \lor k^{-1}(b))) \\
= k(p) \land ((k(p)) \perp \lor b) \\
= f_{k(p)}(b)
\]
Translation $\mathbf{U} : \mathcal{Q} \rightarrow \mathcal{L}$ from algebra to lattice

**on objects**

$\mathbf{U}$ maps an ODA to the orthomodular lattice it induces: $\mathbf{U}(\mathcal{Q}) = (\mathcal{P}_\mathcal{Q}, \preceq, \sim)$.

**on morphisms**

$\mathbf{U}$ maps each morphism to its restriction to $\mathcal{P}_\mathcal{Q}$: if $\zeta : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$, then $\mathbf{U}(\zeta) = \zeta|_{\mathcal{P}_\mathcal{Q}}$. 
The functors $F \circ U$ and $U \circ F$

The elements of $(F \circ U)(\mathcal{Q})$ are

\[ \{ \{ f_{a_1} \circ \cdots \circ f_{a_n} | a_1, \ldots, a_n \in X, n \in \mathbb{N} \} | X \subseteq \mathcal{T}_\mathcal{Q} \} \]

If $\zeta : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2$ is a $\mathcal{Q}$-morphism, then

\[ (F \circ U)(\zeta)(\{ f_{a_1} \circ \cdots \circ f_{a_n} | a_1, \ldots, a_n \in X, n \in \mathbb{N} \}) = \{ f_{\zeta(a_1)} \circ \cdots \circ f_{\zeta(a_n)} | a_1, \ldots, a_n \in X, n \in \mathbb{N} \} \]

The elements of $(U \circ F)(\mathcal{L})$

\[ \{ \{ f_p \} | p \in L \} \]

If $k : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is a $\mathcal{L}$-morphism, then

\[ (F \circ U)(k)(\{ f_p \}) = \{ f_{k(p)} \} \]
The natural isomorphisms

$\eta : 1_\mathcal{Q} \rightarrow F \circ U$

Let $\mathcal{Q}$ be an ODA. Then

$$\eta_\mathcal{Q} : \left( \bigsqcup_{i \in I} a_{i,1} \cdot \cdots \cdot a_{i,n_i} \right) \mapsto \{ f_{a_{i,1}} \circ \cdots \circ f_{a_{i,n_i}} \}_{i \in I}.$$ 

$\tau : 1_\mathcal{L}_b \rightarrow U \circ F$

Let $\mathcal{L}$ be a lattice in $\mathbb{L}$, then

$$\tau_\mathcal{L} : a \mapsto \{ f_a \}$$
Conclusion and future work

- **Connect quantales to quantum structures:** Showed what conditions can be placed on a complemented quantale (orthomodular dynamic algebra) to be categorically equivalent to a complete orthomodular lattice.
- **Future work:** is this the right definition of an ODA?
  - Can weaker morphisms be used?
  - Rather then sets of functions, consider relations instead
- **Future work:** involve unitary operations
- **Future work:** establish a clearer connection to operator algebras
- **Future work:** develop modules for ODA’s to act upon
- **Future work:** develop a logic on ODA’s and compare it to logics on lattices they are equivalent to.

Thank you!
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