Enriched Lawvere Theories for Operational Semantics

John C. Baez
Christian Williams

University of California, Riverside

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How do we integrate syntax and semantics?

object  type
morphism term
* 2-morphism rewrite *
algebraic theories: denotational semantics

\[(ab)c = a(bc)\]

enriched theories: operational semantics
**Lawvere theories**

**Th(Mon)**

- **type**: $M$ monoid

- **operations**
  - $m: M^2 \to M$ multiplication
  - $e: 1 \to M$ identity

- **equations**

```
\begin{align*}
M^3 & \to M^2 \\
\downarrow & \downarrow \\
M^2 & \to M^2 \\
\downarrow & \downarrow \\
M^2 & \to M \\
\downarrow & \\
1 \times M & \to M \\
\downarrow & \\
M & \to M \\
\downarrow & \\
M & \to M \times 1
\end{align*}
```
Enriched theories

\[ \text{Th}(\text{PsMon}) \]

**type**

- M pseudomonoid

**operations**

- \( \otimes : M^2 \to M \) multiplication
- \( \text{id} : 1 \to M \) identity

**rewrites**

\[
\begin{array}{c}
M^3 \xrightarrow{\alpha} M^2 \\
M^2 \xrightarrow{\lambda} M^2 \\
1 \times M \xrightarrow{\lambda} M \\
M \xrightarrow{\rho} M^2 \\
M \times 1 \xleftarrow{\rho} M^2 \\
\end{array}
\]

**equations**

- pentagon, triangle identities
Enriched categories

Let $V$ be monoidal. A $V$-enriched category has hom-objects in $V$; composition and identity are morphisms in $V$, as are the components of a $V$-functor and a $V$-natural transformation:

- **$V$-category** $C(a, b) \in V$
- **$V$-functor** $F_{ab} : C(a, b) \to D(F(a), F(b)) \in V$
- **$V$-transformation** $\varphi_a : 1_V \to D(F(a), G(a)) \in V$.

These form the 2-category $V$Cat.
Our enriching category

Let \( V \) be a cartesian closed category:

\[
V(a \times b, c) \cong V(a, [b, c]).
\]

Then \( V \in V\text{Cat}. \)

Let \( V \in \text{CCC}_{fc(1)} \), meaning assume and choose:

\[
n_V := \sum_n 1_V.
\]

Let \( N_V := \{n_V | n \in \mathbb{N}\} \subset_{\text{full}} V \)

and \( A_V := N_V^{\text{op}} \) — our “arities”. 
Enriched products

The **V-product** of \((a_i) \in C\) is an object \(\prod_i a_i \in C\) equipped with a **V-natural isomorphism**

\[
C(\dash, \prod_i a_i) \cong \prod_i C(\dash, a_i).
\]

A **V-functor** \(F : C \to D\) **preserves** V-products if the “projections” induce a V-natural isomorphism:

\[
D(\dash, F(\prod_i a_i)) \cong \prod_i D(\dash, F(a_i)).
\]

Let \(\text{VCat}_{fp}\) be the 2-category of V-categories with finite V-products and V-functors preserving them.
Enriched Lawvere theories

Definition

A *V-theory* is a $V$-category $T \in V\text{Cat}_{fp}$ whose objects are finite $V$-products of a distinguished object.

A morphism of V-theories is a $V$-functor $F : T \to T' \in V\text{Cat}_{fp}$. These and $V$-natural transformations form the 2-category of $V$-theories, $V\text{Law}$.
Enriched models

Definition

A **context** is a $V$-category $C \in VCat_{fp}$.

A **model** of $T$ is a $V$-functor

$$\mu : T \rightarrow C \in VCat_{fp}.$$ 

The category of models is $Mod(T, C) := VCat_{fp}(T, C)$. 
Example: monoidal categories

Let \( V = \text{Cat} \).

\[
\text{Th}(\text{PsMon})
\]

**type** \( M \) pseudomonoid

**operations**

\( \otimes : M^2 \rightarrow M \) multiplication

\( I : 1 \rightarrow M \) identity

**rewrites**

\[
\begin{align*}
\triangleleft \ & \downarrow \alpha \\
M^3 & \rightarrow M^2 \\
\downarrow & \\
M^2 & \rightarrow M
\end{align*}
\]

\[
\begin{align*}
\uparrow \lambda & \ \\
1 \times M & \rightarrow M^2 \\
& \rightarrow M \\
& \leftarrow M^2
\end{align*}
\]

\[
\begin{align*}
\uparrow \rho & \\
M & \leftarrow M^2 \\
& \rightarrow M \times 1
\end{align*}
\]

**equations** pentagon, triangle identities
## Example: cartesian object

Let $V = \text{Cat}$. 

\[
\begin{array}{llll}
\text{type} & X & \text{cartesian object} \\
\text{operations} & m: X^2 \to X & \text{product} \\
 & e: 1 \to X & \text{terminal element} \\
\text{rewrites} & \Delta: \text{id}_X \Rightarrow m \circ \Delta_X & \text{unit of } m \vdash \Delta_X \\
 & \pi: \Delta_X \circ m \Rightarrow \text{id}_{X^2} & \text{counit of } m \vdash \Delta_X \\
 & \top: \text{id}_X \Rightarrow e \circ !_X & \text{unit of } e \vdash !_X \\
 & \epsilon: !_X \circ e \Rightarrow \text{id}_1 & \text{counit of } e \vdash !_X \\
\text{equations} & \text{triangle identities} \\
\end{array}
\]
Change of base

Let $F : V \to W$ preserve finite products, and $C \in V\text{Cat}$.

Then $F$ induces a change of base:

$$F_*(C)(a, b) := F(C(a, b)).$$

This gives a 2-functor

$$F_* : V\text{Cat} \to W\text{Cat}.$$

Enrichment provides semantics, so change of base should preserve theories to be a change of semantics.
Preservation of theories

**Theorem**

Let \( F : V \to W \in \text{CCC}_{fc(1)} \).

Then \( F \) is a **change of semantics**:

\[ \tau_W := A_W \sim F^*(A_V) \xrightarrow{F^*(\tau_V)} F^*(T) \text{ is a } W\text{-theory.} \]

\( F^* \) preserves models. For every model \( \mu : T \to C \),

\[ F^*(\mu) : F^*(T) \to F^*(C) \text{ is a model of } (F^*(T), \tau_W). \]
Change of semantics

There is a “spectrum” of semantics:

\[
\begin{array}{cccc}
\text{Gph} & \text{Cat} & \text{Pos} & \text{Set} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{FC} & \text{FP} & \text{FS} & \text{UG} \quad \text{UC} \quad \text{UP} \\
\end{array}
\]

- \(\text{FC}_*\) maps small-step to big-step operational semantics.
- \(\text{FP}_*\) maps big-step to full-step operational semantics.
- \(\text{FS}_*\) maps full-step to denotational semantics.
The theory of SKI

\[
\text{Th}(\text{SKI})
\]

<table>
<thead>
<tr>
<th>type</th>
<th>( t )</th>
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<table>
<thead>
<tr>
<th>terms</th>
<th>( S : 1 \to t )</th>
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<tr>
<td></td>
<td>( K : 1 \to t )</td>
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<td></td>
<td>( I : 1 \to t )</td>
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<td></td>
<td>((_ - _): t^2 \to t)</td>
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<tr>
<th>rewrites</th>
<th>( \sigma : (((S \ a) \ b) \ c) \Rightarrow ((a \ c) \ (b \ c)) )</th>
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<tr>
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<td>( \kappa : ((K \ a) \ b) \Rightarrow a )</td>
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<tr>
<td></td>
<td>( \iota : (I \ a) \Rightarrow a )</td>
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A model of Th(SKI)

A Gph-product preserving Gph-functor $\mu : \text{Th}(\text{SKI}) \to \text{Gph}$ yields a graph $\mu(t)$ of SKI-terms:

$$1 \cong \mu(1) \xrightarrow{\mu(S)} \mu(t) \xleftarrow{\mu(\neg \neg)} \mu(t^2) \cong \mu(t)^2.$$  

The rewrites are transferred by the enrichment of $\mu$:

$$\mu_{1,t} : \text{Th}(\text{SKI})(1, t) \to \text{Gph}(1, \mu(t)).$$
The free model of SKI

The syntax and semantics of the SKI combinator calculus are given by the free model

\[ \mu^{Gph}_{SKI} := \text{Th}(SKI)(1, -): \text{Th}(SKI) \to Gph. \]

The graph \( \mu^{Gph}_{SKI}(t) \) is the transition system which represents the small-step operational semantics of the SKI-calculus:

\[ (\mu(a) \to \mu(b) \in \mu^{Gph}_{SKI}(t)) \iff (a \Rightarrow b \in \text{Th}(SKI)(1, t)). \]
Change of semantics

FC: Gph → Cat preserves products, hence gives a change of semantics from small-step to big-step operational semantics:

\[
\begin{align*}
(((S \mathcal{K})(I \mathcal{K})) S) \\
(((S \mathcal{K}) K) S) \\
((K S)(K S))
\end{align*}
\]

FP: Cat → Pos gives full-step (Hasse diagram), and
FS: Pos → Set gives denotational semantics, collapsing the connected component to a point.
Conclusion

Enriched theories give a way to unify the structure and behavior of formal languages.

Enriching in category-like structures reifies operational semantics by incorporating rewrites between terms.

Cartesian functors between enriching categories induce change-of-semantics functors between categories of models.
This paper builds on the ideas of Mike Stay and Greg Meredith presented in “Representing operational semantics with enriched Lawvere theories”.

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References I


References II


M. Stay and L. G. Meredith, Representing operational semantics with enriched Lawvere theories.