The tricategory of formal composites and its strictification

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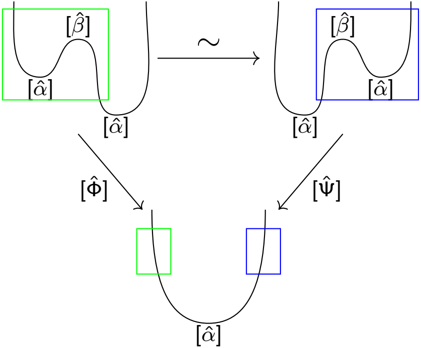
September 11, 2019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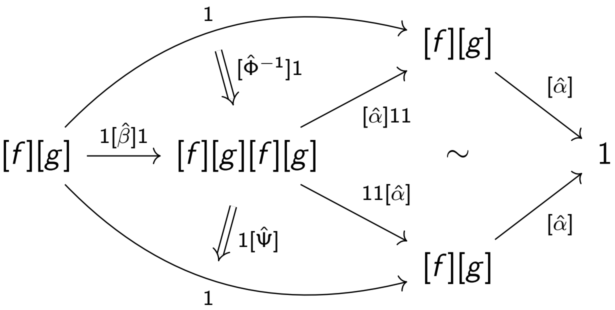
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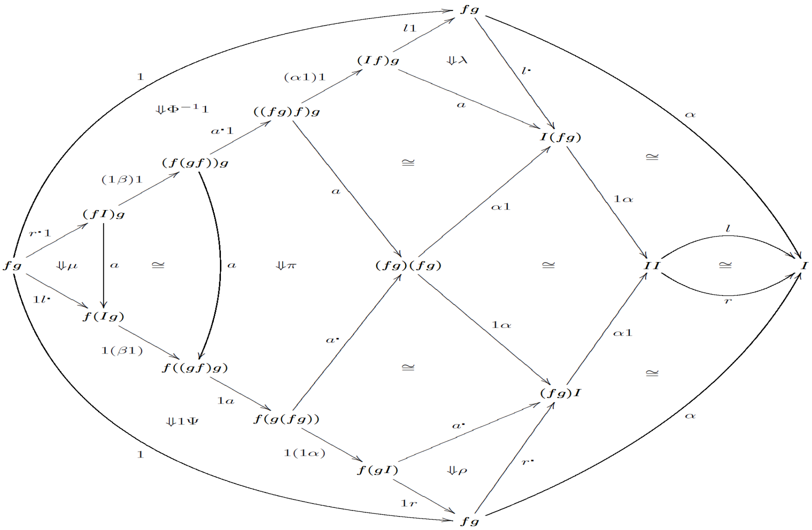
fg
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$f((gf)g)$
$(fg)I$
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$f(g(fg))$
$a^{\bullet}$
1
$1(1\alpha)$
$f(gI)$
$\Downarrow\rho$
$r^{\bullet}$
$1r$
fg

Theorem (Gordon-Power-Street)

Every tricategory $\mathtt{T}$ *is triequivalent to a Gray category* $\mathrm{Gr}\,\mathtt{T}$.

But the triequivalence $\mathtt{T} \to \mathrm{Gr}\,\mathtt{T}$ is not strict.

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But the triequivalence $\mathtt{T} \to \mathrm{Gr}\,\mathtt{T}$ is not strict.

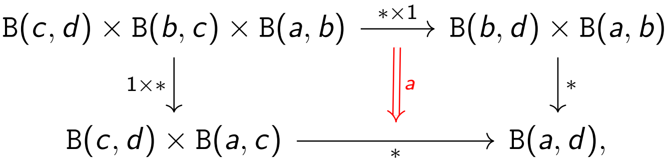
Theorem (G.)

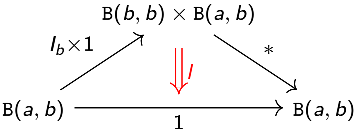
There exists a span of strict triequivalences

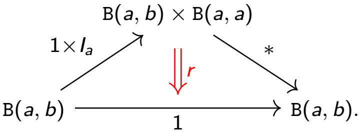
$$\mathtt{T} \xleftarrow{\mathsf{ev}} \widehat{\mathtt{T}} \xrightarrow{[-]} \mathtt{T}^{\mathtt{st}}.$$

Definition (bicategory)

- Collection of objects $\mathrm{Ob}(\mathtt{B})$,
- local hom-categories $\mathtt{B}(a,b)$ for all objects $a,b \in \mathtt{B}$,
- identity functors $I_a : 1 \to \mathtt{B}(a,a)$,
- composition functors $*_{a,b,c} : \mathtt{B}(b,c) \times \mathtt{B}(a,b) \to \mathtt{B}(a,c)$,
- and natural transformations a, l, r corresponding to the axioms of a category.


B(c, d) × B(b, c) × B(a, b)
∗×1
B(b, d) × B(a, b)
1×∗
a
∗
B(c, d) × B(a, c)
∗
B(a, d),


B(b, b) × B(a, b)
I_b×1
l
∗
B(a, b)
1
B(a, b)


B(a, b) × B(a, a)
1×I_a
r
∗
B(a, b)
1
B(a, b).

The axioms of bicategory are chosen such that a coherence law holds.

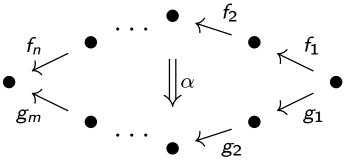
Proposition (Coherence law)

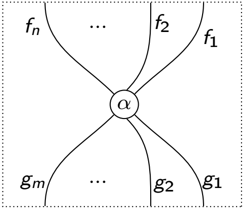
Parallel coherence-morphisms in a free bicategory are equal. (Free on a $\mathtt{Cat}-$ *graph.)*

Definition

A 2-category is called strict if a, l, r are identity natural transformations

In a strict 2-category we can denote 2-cells as follows.

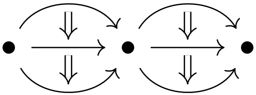




Proposition (Power)

Pasting diagrams are well defined for 2-categories. (And thus string diagrams are.)

Example: interchange law



Folklore 'theorem': Pasting diagrams / String diagrams work also in bicategories. Fix a source fix a target insert coherence cells as needed and the resulting 2-cell will be well defined. I.e. independent of the choice of inserted constraint cells. How

can this made precise?

Proposition

There exists a bicategory $\widehat{\mathrm{B}}$ and a strict 2-category B^{st} together with strict biequivalences as in the following diagram.

$$\mathrm{B} \xleftarrow{\text{ev}} \widehat{\mathrm{B}} \xrightarrow{[-]} \mathrm{B}^{\mathrm{st}}.$$

How does $\widehat{\mathrm{B}}$ look like?

Definition

Let B be a bicategory. Then the following defines a bicategory $\widehat{\mathrm{B}}$ together with a strict biequivalence $\mathrm{ev} : \widehat{\mathrm{B}} \to \mathrm{B}$:

- $\mathrm{Ob}(\widehat{\mathrm{B}}) = \mathrm{Ob}(\mathrm{B})$ and ev acts on objects as an identity.
- The 1-morphisms of $\widehat{\mathrm{B}}$ are formal composites of 1-morphisms in B.
  Thus a generic 2-morphisms looks like:
  $f \,\hat{*}\, \big((g \,\hat{*}\, h) \,\hat{*}\, (k \,\hat{*}\, l)\big)$.
- The action of ev on 1-morphisms is given by evaluation. For example
  $\mathrm{ev}\,\big(f \,\hat{*}\, ((g \,\hat{*}\, h) \,\hat{*}\, (k \,\hat{*}\, l))\big) = f * \big((g * h) * (k * l)\big)$.

Definition

- The 2-morphisms of $\widehat{\mathrm{B}}$ are triples $(\alpha, \hat{f}, \hat{g}) : \hat{f} \to \hat{g}$ where $\alpha : \mathrm{ev}\,\hat{f} \to \mathrm{ev}\,\hat{g}$ is a 2-morphism in B.
- ev acts on 2-morphisms via $\mathrm{ev}(\alpha, \hat{f}, \hat{g}) = \alpha$.
- The constraint-cells of $\widehat{\mathrm{B}}$ are given by

$$\hat{a}_{\hat{f}\hat{g}\hat{h}} = \left(a_{\mathrm{ev}(\hat{f})\,\mathrm{ev}(\hat{g})\,\mathrm{ev}(\hat{h})}, (\hat{f} \,\hat{*}\, \hat{g}) \,\hat{*}\, \hat{h}, \hat{f} \,\hat{*}\, (\hat{g} \,\hat{*}\, \hat{h})\right)$$

$$\hat{l}_{\hat{f}} = (l_{\mathrm{ev}(\hat{f})}, \hat{1}_{t\hat{f}} \,\hat{*}\, \hat{f}, \hat{f}) \quad \text{and} \quad \hat{r}_{\hat{f}} = (r_{\mathrm{ev}(\hat{f})}, \hat{f} \,\hat{*}\, \hat{1}_{s\hat{f}}, \hat{f})$$

- Parallel coherence morphisms in $\widehat{\mathtt{B}}$ are equal.
- Thus coherence can be quotient out of $\widehat{\mathtt{B}}$ which leads to a 2-category $\mathtt{B}^{\mathtt{st}}$.
- Taking equivalence classes gives the desired strict biequivalence $[-] : \mathtt{B} \to \mathtt{B}^{\mathtt{st}}$.

How can the span of strict biequivalences

$$B \xleftarrow{\text{ev}} \widehat{B} \xrightarrow{[-]} B^{st}.$$

be used to reduce calculations in bicategories to calculations in 2-categories.

Example: Adjunction in bicategory $\mathbb{B}$

Data:
1-morphism $f : a \to b$ and $g : b \to a$
2-morphism $\eta : 1_a \to g * f$ and $\epsilon : f * g \to 1_b$.

Axioms:

$$f \xrightarrow{r^{-1}} f * 1 \xrightarrow{1*\eta} f * (g * f) \xrightarrow{a^{-1}} (f * g) * f \xrightarrow{\epsilon*1} 1 * f \xrightarrow{l} f = f \xrightarrow{1} f \tag{1}$$

$$g \xrightarrow{l^{-1}} 1 * g \xrightarrow{\eta*1} (g * f) * g \xrightarrow{a} g * (f * g) \xrightarrow{1*\epsilon} g * 1 \xrightarrow{r} g = g \xrightarrow{1} g. \tag{2}$$

Lift of the adjunction along $\mathrm{ev} : \widehat{\mathtt{B}} \to \mathtt{B}$:

Data:
1-morphism $f : a \to b$ and $g : b \to a$
2-morphism $\hat{\eta} := (\eta, 1_a, g \mathbin{\hat{*}} f)$ and $\hat{\epsilon} := (\epsilon, f \mathbin{\hat{*}} g, 1_b)$

Lift of the adjunction along $\mathrm{ev} : \widehat{\mathtt{B}} \to \mathtt{B}$:

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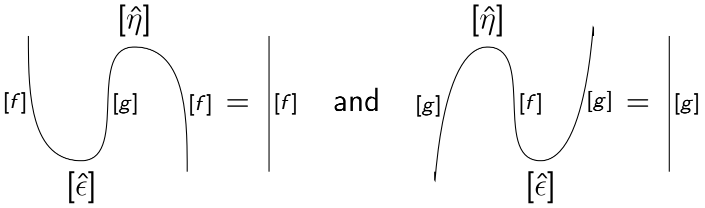
Axioms:

$$f \xrightarrow{\hat{r}^{-1}} f \mathbin{\hat{*}} 1 \xrightarrow{\hat{1} \hat{*} \hat{\eta}} f \mathbin{\hat{*}} (g \mathbin{\hat{*}} f) \xrightarrow{\hat{a}^{-1}} (f \mathbin{\hat{*}} g) \mathbin{\hat{*}} f \xrightarrow{\hat{\epsilon} \hat{*} \hat{1}} 1 \mathbin{\hat{*}} f \xrightarrow{\hat{l}} f = f \xrightarrow{\hat{1}} f \tag{3}$$

$$g \xrightarrow{\hat{l}^{-1}} 1 \mathbin{\hat{*}} g \xrightarrow{\hat{\eta} \hat{*} \hat{1}} (g \mathbin{\hat{*}} f) \mathbin{\hat{*}} g \xrightarrow{\hat{a}} g \mathbin{\hat{*}} (f \mathbin{\hat{*}} g) \xrightarrow{\hat{1} \hat{*} \hat{\epsilon}} g \mathbin{\hat{*}} 1 \xrightarrow{\hat{r}} g = g \xrightarrow{\hat{1}} g. \tag{4}$$

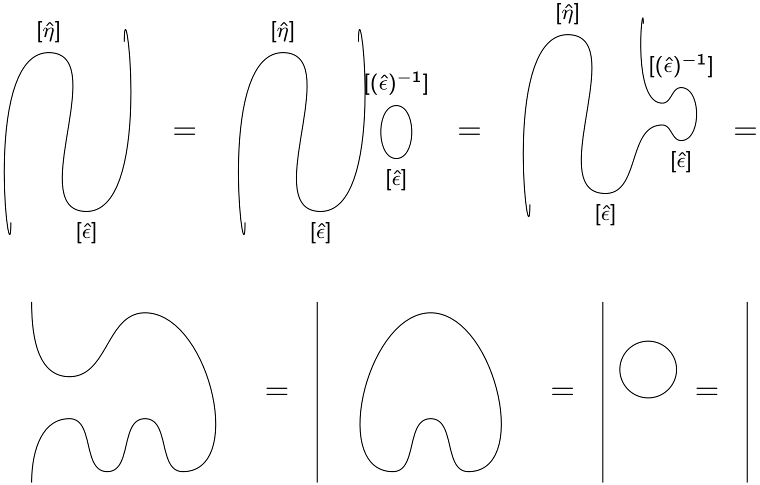
The adjunction in $\widehat{\mathtt{B}}$ under $[-] : \widehat{\mathtt{B}} \to \mathtt{B}^{\mathtt{st}}$:



(5)

Lemma

Let (f, g, η, ϵ) be an equivalence in a bicategory $\mathbb{B}$. Then the equivalence (f, g, η, ϵ) satisfies both equations 1 and 2 if it satisfies one of it.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