Bundles, Lenses & Machine Learning

Jules Hedges[1]
joint work with
Brendan Fong[2] Eliana Lorch[3] David Spivak[2]

[1]Max Planck Institute for Mathematics in the Sciences

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SYCO 5, Birmingham

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Featuring zero string diagrams :(

Motivation

Machine learning is categorical in 2 different ways:

Backprop As Functor
(compositional description of ML with monoidal categories)
+
ML as differential geometry

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In this talk: smoosh them together

(why? Why not)

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(why? Why not)

It clarifies Backprop as Functor more than anything else

Open learners

Definition (Fong, Spivak & Tuyéras) : An open learner $X \to Y$ consists of:

- A set P of parameters

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Composition of open learners is fiddly

They form a symmetric monoidal category called **Learn**

who cares about monoidal bicategories

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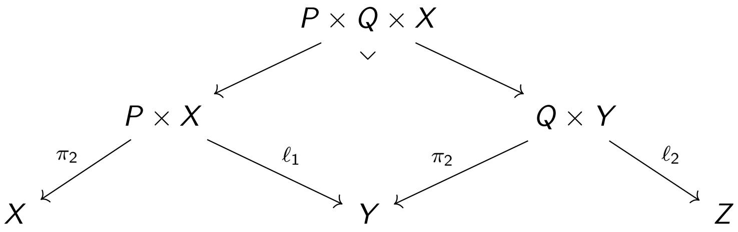
Composition of lenses is also fiddly!

Theorem (Fong & Johnson): Open learners compose by pullback of lenses:



The **Para** construction

Let $\mathcal{C}$ be a monoidal category

Define a category[1] **Para**($\mathcal{C}$) by:

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$$(A \otimes B, X \otimes A \otimes B \xrightarrow{f \otimes B} Y \otimes B \xrightarrow{g} Z)$$

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$\otimes$ lifts to a monoidal product on **Para**($\mathcal{C}$)

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The structure of $\mathbf{Para}(-)$

A lax symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ lifts to

$$\mathbf{Para}(F) : \mathbf{Para}(\mathcal{D}) \to \mathbf{Para}(\mathcal{D})$$

by

$$F(A, f) : F(X) \otimes F(A) \xrightarrow{\varphi} F(X \otimes A) \xrightarrow{F(f)} F(Y)$$

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Proposition (probably): **Para**$(-)$ defines a monad on [symmetric monoidal categories, lax symmetric monoidal functors]

Backprop as Functor

Theorem (Fong, Spivak & Tuyéras): Fix a learning rate $\varepsilon > 0$ and a differentiable cost function[2] $C : \mathbb{R}^2 \to \mathbb{R}$.

[2] such that every $\frac{\partial}{\partial y} C(x, y)$ is invertible

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Theorem (Fong, Spivak & Tuyéras): Fix a learning rate $\varepsilon > 0$ and a differentiable cost function[2] $C : \mathbb{R}^2 \to \mathbb{R}$.

Then there is a symmetric monoidal functor $F_{\varepsilon,C} : \mathbf{Para}(\mathbf{Euc}) \to \mathbf{Learn}$ defined by

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Update is gradient descent, and request is backpropagation

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(The benefit is **Learn** is more general than just ML)

Bundles

Work in a category with finite limits

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3. Cotangent bundle $\begin{array}{c} T^*\mathcal{M} \\ \downarrow \pi^* \\ \mathcal{M} \end{array}$

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Nb. **Euc** doesn't have finite limits, so we work in **Top**

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"Every algebraic geometer knows this definition" – David Spivak

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Identity morphism:

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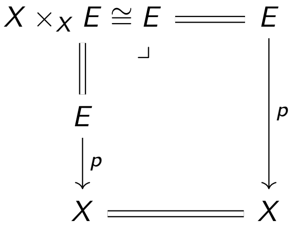
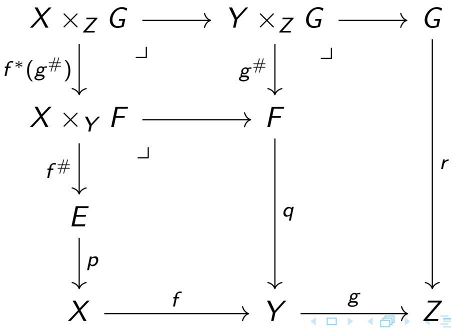


Composition of morphisms:



Where does this come from?

From the Grothendieck construction:

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This buys us (conjecture) a monoidal structure:

$$\begin{array}{c} E \\ \downarrow p \\ X \end{array} \otimes \begin{array}{c} F \\ \downarrow q \\ Y \end{array} = \begin{array}{c} E \times F \\ \downarrow p \times q \\ X \times Y \end{array}$$

(this might not be the right one!)

Lenses

A (bimorphic) lens $\lambda : (S, T) \to (A, B)$ consists of:

- a morphism $\lambda_v : S \to A$ called view
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Theorem (Lambek): $\mathrm{coKl}(X \times -) \cong \mathcal{C}[x]$, where $\mathcal{C}[x]$ is the polynomial category formed by freely adjoining $x : 1 \to X$ and closing under finite products

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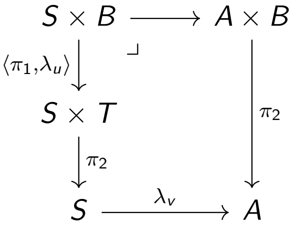
Grothendieck them all together: $\mathbf{Lens}(\mathcal{C}) \to \mathbf{Bund}(\mathcal{C})$
It takes a lens $\lambda : (S, T) \to (A, B)$ to the bundle morphism



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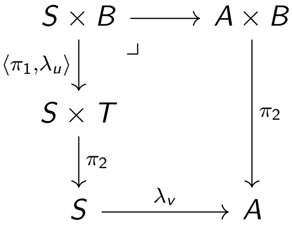

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$J_x(f)$ is the Jacobian (matrix of partial derivatives) of f at x

The chain rule

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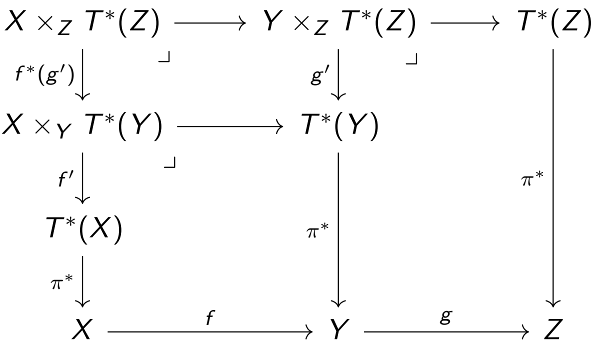

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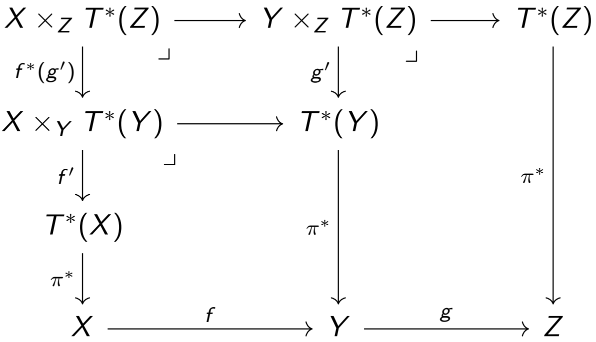

$(g \circ f)' = f' \circ f^*(g')$ is the chain rule in differential geometry

From **Para(Bund(Top))** to **Learn**

Consider a morphism of **Para**(**Bund**(**Top**)) in the image of

$$\mathbf{Para}(\mathbf{Cot}) : \mathbf{Para}(\mathbf{Euc}) \to \mathbf{Para}(\mathbf{Bund}(\mathbf{Top}))$$

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We're going to turn it into an open learner, given $\varepsilon > 0$ and differentiable $C : \mathbb{R}^2 \to \mathbb{R}$

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so, fix $a \in A$, $x \in X$ and $y \in Y$

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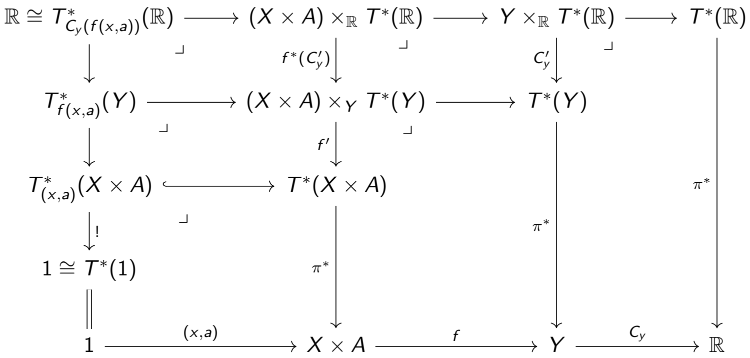
Consider the diagram. . .

The brain exploding part



The part we don't understand

Now: Chase $1 \in \mathbb{R}$ to $T^*(X \times A)$ and then apply

$$\mu_\varepsilon : T^*(X \times A) \to X \times A$$

The result is $\langle r, U \rangle (a, x, y)$

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It looks a bit like a thing called an exponential map

The catch

Conjecture: This defines a symmetric monoidal functor

$$\mathbf{Para}(\mathbf{Bund}(\mathbf{Top})) \supseteq \mathrm{Im}(\mathbf{Para}(\mathbf{Cot})) \to \mathbf{Learn}$$

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So, we've just rewritten Backprop as Functor in a different way!

Even more hard questions

What happens if we extend the functor to the whole of **Para**(**Bund**(**Top**))? We have no idea!

Optimistic hope: This allows defining general "ML-like" systems, not necessarily involving gradients (eg. "discrete ML" on Bayesian networks)