The Dialectica category & related structures

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Motivation: Gödel's Dialectica interpretation, 1958

→ categorical model for Linear Logic (classical/multimodal)

Girard's Linear Logic, ~1986 (linear/symmetric)

→ decompose logical connectives: ⇒ to ⊃,
  ! (of course)

- Lenses (asymmetric/multimorphic, 2003 Pierre/Schmitt,
  2007 + Foster, Greenwald, Moore)

Motivation: modeling box transformations

view-update problem (database theory since late '70)

- Wiring diagrams (Spivak, 2013
  + Rupel, Vagner, Lerman, Schultz, C'),...)

Motivation: systems as operad algebras

compositional analysis (zoom in/out, re-design)

Moore machines, continuous dynamical systems,

abstract machines...

- Relaxed: open game theory (Hedges, Ghani, 2015)

  → learners (Fong, Spivak, Myers, 2017)

→ 2017 Hyland: WD + Dialectica

2018 ACT Leiden: Lenses + WD (CV + joint project)
Wiring diagrams: categorical formalism for pictures like

Every possible interconnection of "boxes" is a morphism in a symmetric monoidal category (orthogonal to usual string theory pics!)

- For any $\mathcal{C}$, the category $\text{WC}$ of labelled boxes & wiring diagrams has
  - objects pairs $(X, X')$ of $\mathcal{C}$-typed finite sets, e.g. $\text{Set}$ - labelled box
    $$\begin{array}{c}
    \text{input} \\
    \{a_1, \ldots, a_m\} \quad \text{output} \\
    \{b_1, \ldots, b_n\}
    \end{array}
    \quad \overset{\tau}{\longrightarrow}
    \quad \begin{array}{c}
    \{a_1, \ldots, a_m\} \quad \text{input} \\
    \{b_1, \ldots, b_n\} \quad \text{output}
    \end{array}
    $$
    \begin{align*}
    \tau(a_i) &= \mathbb{Z} \\
    \tau(a_0) &= \{1, \tau\}
    \end{align*}

- morphisms $(X, X') \to (Y, Y')$ pairs of functions $X \to X_2 + Y_1$ that respect the type

- monoidal structure

```
X + Y1  \\
X2 + Y2
```

```
\theta = \theta
```
When $G$ has products, this "maps" to a category $\overrightarrow{W_G}

\text{[idea: associate objects of } G \text{ rather than set to input & output sides of boxes!]}$

- objects are pairs $(S = \prod_{x, y} x, T = \prod_{x, y} y) \in G \times G$
  - product of all output types
  - e.g. $Z \times \{1, 2\}, T \times \ldots \times NV \in \mathbb{E}$

- morphisms $(S, T) \to (A, B)$ are
  - "taking products" is (contravariant & strongly monoidal)
  - summarized into product

\[
\begin{array}{c}
\text{Composition: } S \times T \Rightarrow (A, B) \\
\begin{array}{c}
S \\
\times \downarrow \downarrow \downarrow \\
A \\
\times \uparrow \uparrow \uparrow \\
B
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{monoidal structure} \\
\begin{array}{c}
\otimes \\
\otimes \downarrow \downarrow \downarrow \\
\odot \\
\otimes \uparrow \uparrow \uparrow \\
\bowtie
\end{array}
\end{array}
\]

- $\Rightarrow (\overrightarrow{W_G}, \otimes, \odot)$ is a symmetric monoidal category

Various systems are algebras on $\overrightarrow{W_G}$ (or the induced operad)

i.e. box monoidal (pseudo) functors $K: (\overrightarrow{W_G}, \otimes, \odot) \to (G_x, \times, 1)$

$K(S, T) = \text{category of all Moore machines}$

$A \times 0 \text{ with input } T \text{ & output } S$

functionality & monoidality of $K$

produces a NEW $M$ as their composite!

... completely expressed in terms of subsystems
**Dialectica categories**

Take \( C \) to be monoidal closed with products, \((\otimes, \otimes, I, \rightarrow, \times)\)

- \( C \times C^{\text{op}} \) has monoidal structure

\[
(s, t) \otimes (a, b) := (s \otimes a, (a \rightarrow t) \times (s \rightarrow b))
\]

In fact, it's closed:

\[
(s \otimes a, (a \rightarrow t) \times (s \rightarrow b)) \rightarrow (p, r)
\]

\[
(s, t) \rightarrow ((a \rightarrow p) \times (r \rightarrow b), r \otimes a)
\]

\( G(C) = C \times C^{\text{op}} \) is a categorical model of **Classical Linear Logic**

- In particular, it has products \((s \times a, t \times b)\)
- and coproducts \((s + a, t + b)\)
- Moreover, co-products

Suppose \( C \) is ccc.

- There is a comonad \( F : C \times C^{\text{op}} \rightarrow C \times C^{\text{op}} \)

- **Comultiplication:** \( F(s, t) \rightarrow F(F(s, t)) \in C \times C^{\text{op}} \)

\[
(s, t) \rightarrow (s', t') \rightarrow (s'' \times s''', s'' \times t''')
\]

- **Counit:** \( F(s, t) \rightarrow (s, t) \)

\[
Id_s : s \rightarrow s
\]

\[
0 : t \rightarrow s \times t
\]

\[
\Rightarrow : s \times t \rightarrow t
\]

- The Dialectica category \( D(G) \) is the coKleini category \((C \times C^{\text{op}})_F\)
- It has the same objects \((s, t)\) and morphisms \((s, t) \rightarrow (a, b)\) are \( F(s, t) = (s, t') \rightarrow (a, b) \in \ C \times C^{\text{op}} \)
- Namely:

\[
\begin{align*}
S & \rightarrow A \\
B & \rightarrow T
\end{align*}
\]

\[
S \otimes B \rightarrow T
\]

\( D(G) \) is (\( \Rightarrow \)) a categorical model for (the propositional part of)

\[\text{Intuitionistic Linear Logic}\]

weak coproducts

\[
\Rightarrow
\]
Remarks

- $D(-)$ and $W(-)$ are functors:

$$\text{ftCat} \rightarrow \text{SymMonCat}$$

$$t \mapsto W_t$$

$$f \mapsto J_f$$

$$D(f) \mapsto (f, \text{ft})$$

- In original work, $D(G)$ has objects "relations" $S \rightarrow S \times T$ and morphisms $g : S \rightarrow S \times T$ such that a non-trivial condition is satisfied.

- In easier $G(G)$, with $S \rightarrow S \times T$:

$$S \times \Delta(b) \leq g(s) \Delta(b) \quad A + B$$

is a "semi-adjointness" condition.

- Standard colax-kleisli theory

$$\varepsilon : \varepsilon \circ \rho \rightarrow 1$$

$$D(G) = (\varepsilon \circ \rho)$$

E.g., $D(G)$ inherits products via right adjoint $(S \times A, T \times B)$, but no more limits or colimits are expected in general! (weak coadjunction)

Lenses: interactions between a database a view of it

Monomorphic/asymmetric lenses are two objects $S, A$ in $\mathcal{C}$ (product with $S \times A \rightarrow S$ "put" - UPDATE source VIEW

$S \rightarrow A$ "get" - VIEW (e.g. database) (e.g results)

Clearly, this is a wiring diagram/diagramic morphism $(SS) \rightarrow (AA)$
Well-behaved lenses satisfy

Put Get: \( S \times A \xrightarrow{p} S \xrightarrow{\text{Get}} A = \text{id}_A \)

"You get back what you put in!"

Get Put: \( S \xrightarrow{\text{Get}} S \xrightarrow{\text{Put}} S \xrightarrow{\text{Get}} A \xrightarrow{\text{Put}} S = \text{id}_S \)

"Putting back what you got doesn't change anything."

E.g. constant complement view-updating lens

\[
(\mathbb{A}_1 \times \mathbb{A}_2, \mathbb{A}_1)
\]

by

\[
\begin{align*}
\alpha_1 & \quad \alpha_2 \\
\mathbb{A}_1 & \quad \mathbb{A}_2 \\
\text{source view} & \\
\end{align*}
\]

\[
\begin{align*}
\Sigma \text{p}: (\mathbb{A}_1 \times \mathbb{A}_2) \times \mathbb{A}_1 & \xrightarrow{\pi_2} \mathbb{A}_1 \times \mathbb{A}_2 \\
\Sigma \text{g}: \mathbb{A}_1 \times \mathbb{A}_2 & \xrightarrow{\text{id}} \mathbb{A}_1 \\
\end{align*}
\]

is a well-behaved lens [complement remains unchanged]

Bimorphic lenses \((S, T) \rightarrow (A, B)\) are precisely dialectica morphisms

\[
\begin{align*}
S \times B & \rightarrow T \\
S & \rightarrow A \\
\end{align*}
\]

view can change to different type \(B\) resulting to a change of the whole from \(S\) to type \(T\).

\[D(S) \equiv \hat{W}_S \equiv \text{Bilens}(S)\]

Open questions/directions: channel of communication between completely different areas serving different purposes.

- Transfer of structured properties & intuition
  - \((S, S) \rightarrow (A, A)\) monomorphic lenses
  - \((S, S) \rightarrow (A, B)\) Moore machines
  - \((S, T) \rightarrow (A, B)\) Dialectica translation of implication

Classify/understand subcategories of interest

- Conditions for relations in Dialectica & functionality
  - Also, some algebras themselves expressed as Dialectica maps....
  - \(D(S)\) monoidal closedness \([S, T], (A, B)\] = \((A \times T, S \times B)\).
  - What does it mean for RVD bilens?

- Symmetric lenses = 3 pens of asymmetric span-like algebras