Completeness of qubit ZX calculus via elementary operations

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Outline

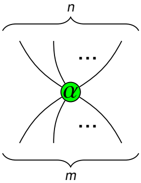
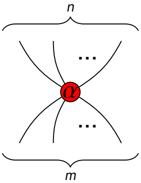
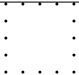
What is ZX-calculus

- ZX-calculus is a graphical language for quantum computing proposed by Coecke and Duncan [ICALP'08, New J. Phys., 2011].
- It gives all the details of interacting processes in quantum computation using qubits.
- ZX-calculus can be formalised in the framework of PROPs, which are strict symmetric monoidal categories having the natural numbers as objects, with the tensor product of objects given by addition.
- As a PROP, ZX-calculus can be presented by generators and relations (rewriting rules), just like the presentation of a group.

How useful is completeness

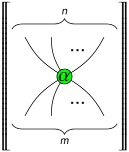
- Completeness of ZX-calculus means quantum computing can be done pure diagrammatically.
- Completeness offers a complete set of rules based on which one could develop an efficient rule set for particular application purpose.
- The key idea of applying ZX-calculus is first encoding matrices into diagrams then choosing suitable rules to rewrite diagrams into a form as simple as you can.

Original generators of ZX-calculus

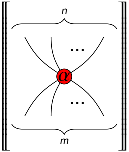
$R_{Z,\alpha}^{(n,m)} : n \to m$		$R_{X,\alpha}^{(n,m)} : n \to m$	
$H : 1 \to 1$		$\sigma : 2 \to 2$	
$\mathbb{I} : 1 \to 1$		$e : 0 \to 0$	
$C_a : 0 \to 2$		$C_u : 2 \to 0$	

where $m, n \in \mathbb{N}$, $\alpha \in [0, 2\pi)$, $a \in \mathbb{C}$, and e represents an empty diagram.

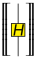
Standard interpretation of ZX-calculus



$$= |0\rangle^{\otimes m}\langle 0|^{\otimes n} + e^{i\alpha}|1\rangle^{\otimes m}\langle 1|^{\otimes n},$$



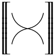
$$= |+\rangle^{\otimes m}\langle +|^{\otimes n} + e^{i\alpha}|-\rangle^{\otimes m}\langle -|^{\otimes n},$$



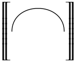
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},$$

$$= 1,$$

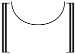
$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$



$$= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$



$$= \begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix},$$



$$= \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix},$$

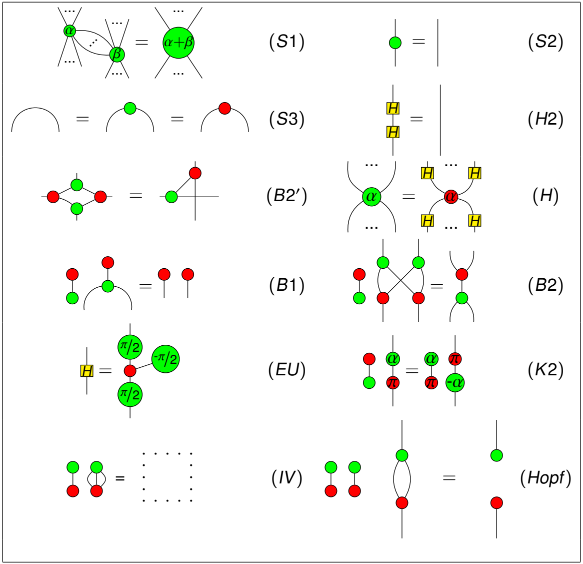
$$[\![D_1 \otimes D_2]\!] = [\![D_1]\!] \otimes [\![D_2]\!], \quad [\![D_1 \circ D_2]\!] = [\![D_1]\!] \circ [\![D_2]\!],$$

where

$$|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \langle 0| = \begin{pmatrix}1 & 0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}, \quad \langle 1| = \begin{pmatrix}0 & 1\end{pmatrix},$$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \quad \langle +| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \quad \langle -| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}.$$

Typical rewriting rules of ZX-calculus

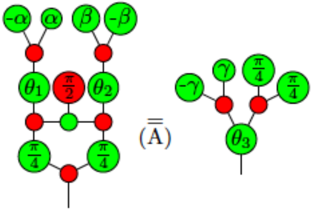


Three properties of the ZX-calculus

- The ZX-calculus is sound: for any two diagrams D_1 and D_2, $ZX \vdash D_1 = D_2$ must imply that $[\![D_1]\!] = [\![D_2]\!]$. [Coecke, Duncan, New J. Phys., 2011]
- The ZX-calculus is universal: for any linear map L, there must exist a diagram D in the ZX-calculus such that $[\![D]\!] = L$. [Coecke, Duncan, New J. Phys., 2011]
- The ZX-calculus is complete: for any two diagrams D_1 and D_2, $[\![D_1]\!] = [\![D_2]\!]$ must imply that $ZX \vdash D_1 = D_2$. [Hadzihasanovic, Ng, Wang, LICS'18; Jeandel, Perdrix, Vilmart, LICS'18]

Why another complete axiomatisation for qubit ZX-calculus

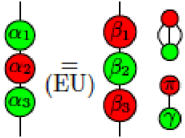
- The following non-linear axiom was presented in [Jeandel, Perdrix, and Vilmart, LICS'18] and [Jeandel, Perdrix, and Vilmart, LICS'19]:



$$2e^{i\theta_3}\cos(\gamma) = e^{i\theta_1}\cos(\alpha) + e^{i\theta_2}\cos(\beta)$$

Why another complete axiomatisation for qubit ZX-calculus

- The following non-linear axiom was presented in [Vilmart, LICS'19]:

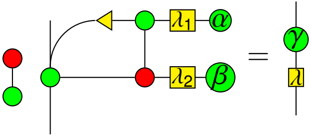


where

$$\begin{cases} x^+ := \frac{\alpha_1+\alpha_3}{2} \qquad x^- := x^+ - \alpha_3 \\ z := \cos\left(\frac{\alpha_2}{2}\right)\cos\left(x^+\right) + i\sin\left(\frac{\alpha_2}{2}\right)\cos\left(x^-\right) \\ z' := \cos\left(\frac{\alpha_2}{2}\right)\sin\left(x^+\right) - i\sin\left(\frac{\alpha_2}{2}\right)\sin\left(x^-\right) \\ \beta_1 = \arg z + \arg z \\ \beta_2 = 2\arg\left(i + \left|\frac{z}{z'}\right|\right) \\ \beta_3 = \arg z - \arg z' \\ \gamma = x^+ - \arg(z) + \frac{\alpha_2-\beta_2}{2} \end{cases}$$

Why another complete axiomatisation for qubit ZX-calculus

- The following non-linear axiom was presented in [Hadzihasanovic, Ng, Wang, LICS'18]:



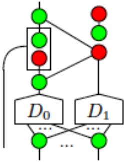
where $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$.

- Except for [Jeandel, Perdrix, and Vilmart, LICS'19], all the other completeness proofs need the translation from the ZW-calculus.
- All these proofs are not easy to generalise to qudit cases.

Normal form by [Jeandel, Perdrix, and Vilmart, LICS'19]

- The normal form used in [Jeandel, Perdrix, and Vilmart, LICS'19] is defined recursively.
- **(Controlled scalars).** A ZX-diagram $D : 1 \to 0$ is a controlled scalar if $[\![D]\!]\,|0\rangle = 1$.
- **(Controlled Normal Form).** Given a set S of controlled scalars, the diagrams in normal controlled form with respect to S (S-CNF) are inductively defined as follows:

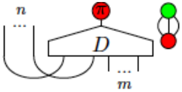
– $\forall D \in S$, D *is in* S-CNF;

– $\forall D_0, D_1$ *in* S-CNF,  *is in* S-CNF.

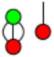
A diagram D *in* S-CNF *is depicted* .

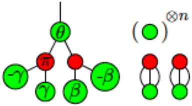
Normal form by [Jeandel, Perdrix, and Vilmart, LICS'19]

- **(Normal Form).** Given a set S of controlled scalars, for any $n, m \in \mathbb{N}$, and any $D : 1 \to n + m$ in S-CNF, the following diagram is called a normal form with respect to S (S-NF):



- Define $\Lambda_{\mathbb{R}} : \mathbb{C} \to ZX[1, 0]$ as:

– $\Lambda_{\mathbb{R}}(0) =$ 

– $\forall \rho > 0, \forall \theta \in [0, 2\pi),\ \Lambda_{\mathbb{R}}(\rho e^{i\theta}) :=$ 

$$\begin{pmatrix} n := \max\left(0, \lceil \log_2(\rho) \rceil\right) \\ \beta := \arccos\left(\frac{\rho}{2^n}\right) \\ \gamma := \arccos\left(\frac{1}{2^n}\right) \end{pmatrix}$$

and $S_{\mathbb{R}} := \{\Lambda_{\mathbb{R}}(x) \mid x \in \mathbb{C}\}$.

- **Theorem** [Jeandel, Perdrix, and Vilmart, LICS'19] Any ZX-diagram can be put into a normal form with respect to$S_{\mathbb{R}}$, and the ZX-calculus is complete for the full pure quibt QM.

Generators for pure linear complete axiomatisation of qubit ZX-calculus

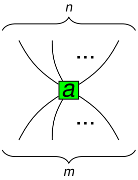
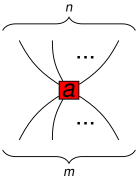
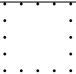
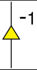
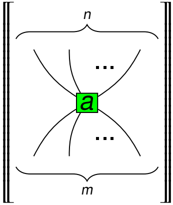
$R_{Z,\alpha}^{(n,m)}: n \to m$		$R_{X,\alpha}^{(n,m)}: n \to m$	
$H: 1 \to 1$		$\sigma: 2 \to 2$	
$\mathbb{I}: 1 \to 1$		$e: 0 \to 0$	
$C_a: 0 \to 2$		$C_u: 2 \to 0$	
$T: 1 \to 1$		$T^{-1}: 1 \to 1$	

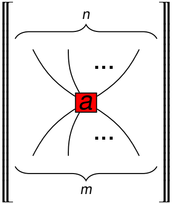
Table: Generators of qubit ZX-calculus

where $m, n \in \mathbb{N}$, $\alpha \in [0, 2\pi)$, $a \in \mathbb{C}$, and e represents an empty diagram.

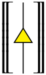
Standard interpretation of new generators

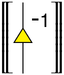


$$= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + a\, |1\rangle^{\otimes m} \langle 1|^{\otimes n},$$



$$= |+\rangle^{\otimes m} \langle +|^{\otimes n} + a\, |-\rangle^{\otimes m} \langle -|^{\otimes n},$$


$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$



$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

where a is an arbitrary complex number.

Rules for pure linear complete axiomatisation of qubit ZX-calculus

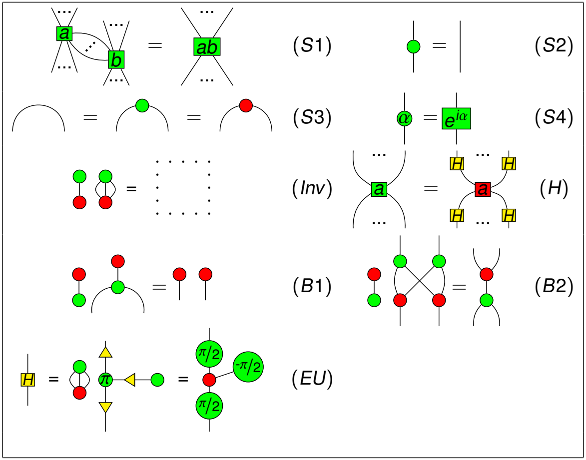


Figure: Rules I, where $\alpha, \beta \in [0,\ 2\pi)$, $a, b \in \mathbb{C}$.

Rules for pure linear complete axiomatisation of qubit ZX-calculus

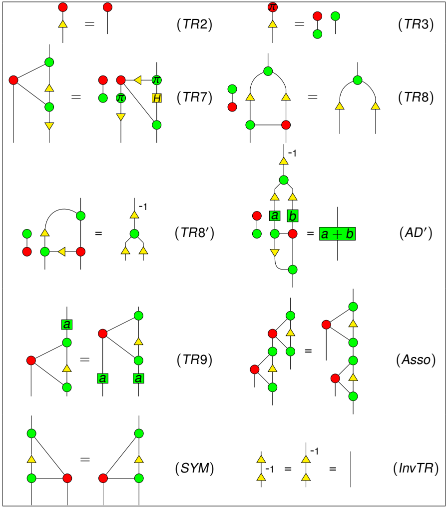


Figure: Ruels II, $a, b \in \mathbb{C}$

Rules for pure linear complete axiomatisation of qubit ZX-calculus

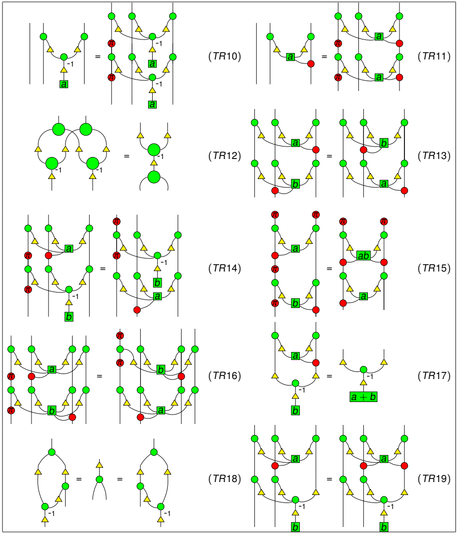
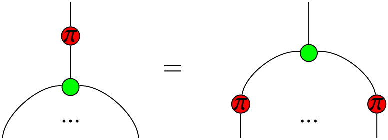
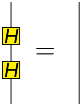


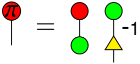
Figure: Ruels III, $a, b \in \mathbb{C}$

Derivable rules

- 

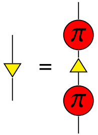


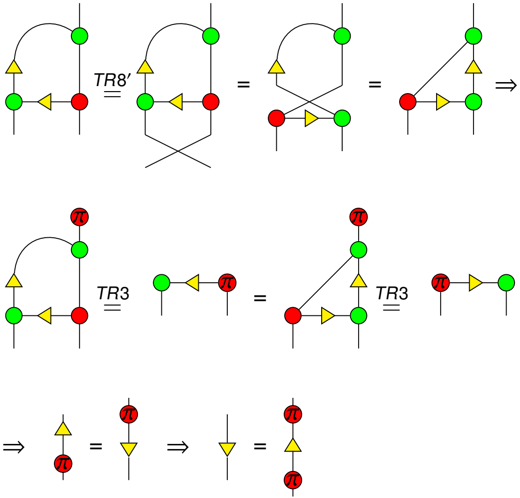
Proved in [Backens, Perdrix, Wang, QPL'16]

- 

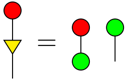
(TR3')
Directly obtained by plugging a triangle on both sides of (TR3).

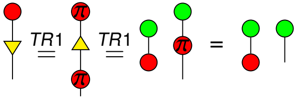
Derivable rules

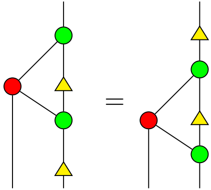
-  (TR1) Proof:

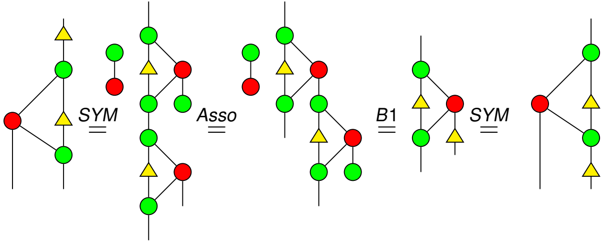


Derivable rules

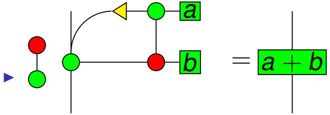
-  (TR2') Proof:



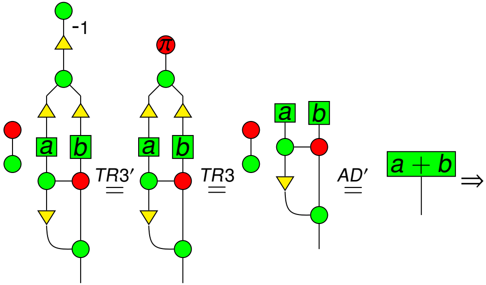
-  Proof:

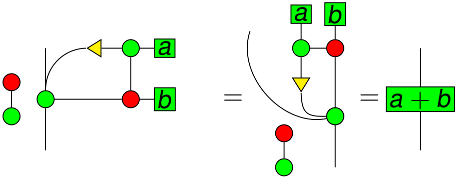


Derivable rules

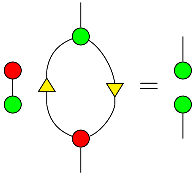


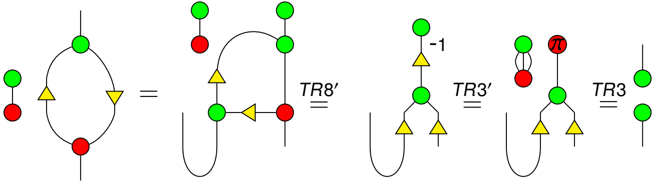
(AD) Proof:



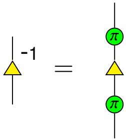


Derivable rules

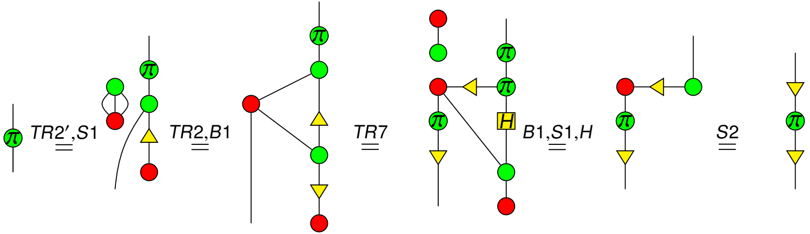
-  (TR4) Proof:



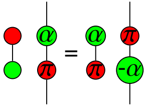
Derivable rules

- 

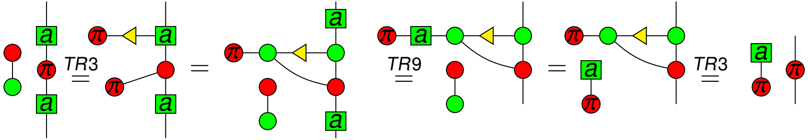
(IVT) Proof:



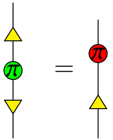
Derivable rules

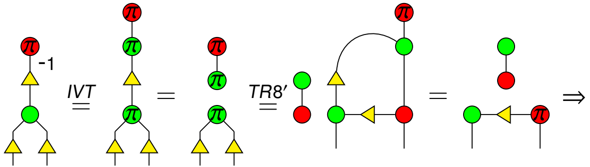
- 

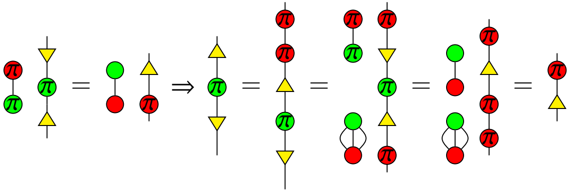
(K2) Proof:



Derivable rules

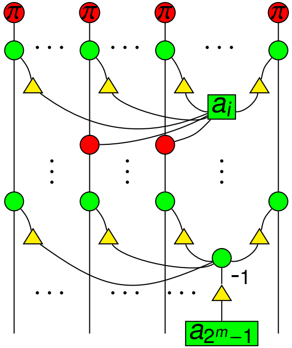
-  (TR5) Proof:





Normal form

Any complex vector $(a_0, a_1, \cdots, a_{2^m-1})^T$ can be uniquely represented by



where a_i connects to wires by red nodes depending on i, and all possible connections are included in the normal form.

Where does this normal form come from

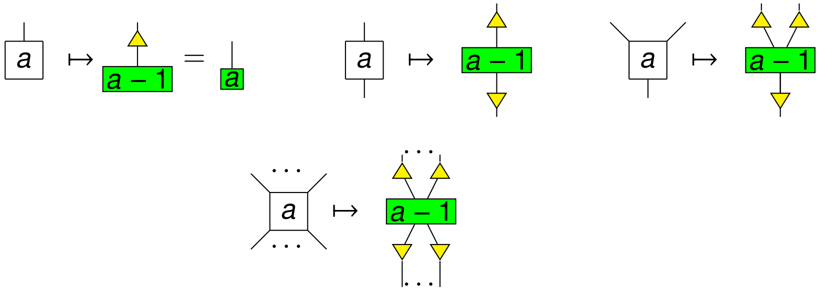
$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \xrightarrow[\text{addition}]{\text{row}} \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \xrightarrow[\text{addition}]{\text{row}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^m-1} \\ 1 \end{pmatrix} \xrightarrow[\text{multiplication}]{\text{row}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^m-2} \\ a_{2^m-1} \end{pmatrix}$$

How to prove completeness

- the juxtaposition of any two diagrams in normal form can be rewritten into a normal form.
- a self-plugging on a diagram in normal form can be rewritten into a normal form.
- all generators can be rewritten into normal forms.

One simple application of the linear version of ZX

- Translate arbitrary H-box in ZH to ZX:



where the general H-box corresponds to the matrix:

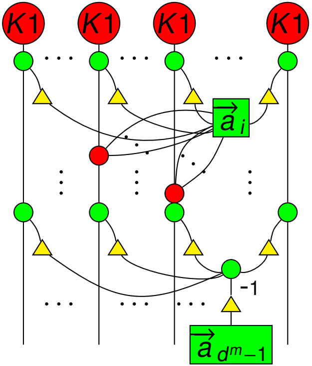
$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & a \end{pmatrix}.$$

- With this translation, we can say that ZH is “SLOCC equivalent” to ZX.

Further work

- Generalise the completeness result of the ZX-calculus from qubit to qudit for arbitrary dimension d. Normal form for

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d^m-2} \\ a_{d^m-1} \end{pmatrix} :$$



.

where

$K_1 = |d-1\rangle, \overrightarrow{a}_i = (0, \cdots, 0, a_i), \overrightarrow{a}_{d^m-1} = (1, \cdots, 1, a_{d^m-1})$.

Further work

- Achieve a complete axiomatization of the ZX-calculus with mixed dimensions.
- Find useful ZX rules for optimisation of Benchmark quantum circuits.
- Apply to linguistics.

Thank you!